Analysis of the entropy vector approach to distinguish classical and quantum causal structures

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The exploration of causal mechanisms is an integral part of science and technology. Causal structures, as the main systematic means towards a fundamental understanding of such mechanisms, are omnipresent in science. For instance, in medicine it is relevant to identify whether a new drug actually cures a disease or whether patients respond equally to placebos; in economics causal insights are relevant to consciously regulate the market; and physicists often make predictions about phenomena in terms of cause and effect.

Bell's theorem shows that our intuitive understanding of causation does not apply to quantum correlations [1]. Nevertheless, quantum mechanics does not permit signalling and hence a notion of cause remains. Understanding this notion is not only conceptually interesting, but also important for technological applications. Ruling out classical common causes is important in information theory [2–10], especially for device-independent cryptography where it has recently been shown that the ability to demonstrate non-classicality implies the ability to generate secure random numbers [6–9]. It is therefore important to characterise the set of classical correlations compatible with different causal structures as far as possible.

A useful way to determine which classical causal structures give rise to a given set of correlations is by using so called entropy vectors [11–17]. We analyse this technique and derive new entropic inequalities that characterise classical causal structures, leading to an improvement on the classical causal inference problem. We furthermore give the first derivation of sufficient conditions for entropy vectors to be achievable in particular causal structures, which, together with our improvement on the necessary conditions, brings us closer to a characterisation of the boundary of the actual cones [18].

We also prove that there are limitations regarding the application of such entropic techniques to causal scenarios that involve quantum systems. The standard approach for employing entropic techniques is not sufficient in this respect and we analyse which techniques may still lead to further insights [19].

Considering causal structures with the entropy vector approach. Classical causal structures may be analysed within the theory of Bayesian networks [20, 21]. Formally, a classical causal structure is considered as a set of nodes that are arranged in a directed acyclic graph (DAG). Each node is associated with a random variable for which the DAG encodes constraints on their joint distribution as follows: Let C be a classical causal structure with nodes $X_1, X_2, \ldots, X_n$, where the first k nodes are observed. An observed joint probability distribution over these k variables is *compatible* with C, if it can be decomposed as

$$P_{\mathrm{X_1 X_2 \dots X_k}} = \sum_{X_{k+1},\dots,X_n} \prod_{i=1}^{n} P_{\mathrm{X_i | X_i^{\downarrow 1}}}, \tag{1}$$

where $X_i^{\downarrow 1}$ denotes the parent nodes in the DAG (i.e., all nodes from which an arrow points directly at X_i).

While the problem of checking compatibility for a certain distribution is in principle solved, it is in practice difficult to check whether a particular distribution can be decomposed according to (1). Furthermore, as most causal structures yield sets of compatible probability distributions that are non-convex, deriving certificates to rule out membership is challenging. The search for simplifications and methods to tackle this problem is an active area of research [11–17, 22–28], where special emphasis has been lying on the use of entropies [11–17, 22, 26].

In these entropic approaches [11, 13, 16, 29], the basic idea is to map the joint distribution $P_{\mathrm{X_1 X_2 \dots X_k}}$ as well as all its marginal distributions to their respective Shannon entropies, which leads to an *entropy vector*

$$(H(X_1),\ H(X_2),\ \dots\ ,\ H(X_k), H(X_1X_2),\ H(X_1X_3),\ \dots\ ,\ H(X_1X_2\ \dots\ X_k)).$$

Any such vector has to obey a set of constraints that are known to hold for the Shannon entropy of any distribution, e.g. the positivity of all its components. The independences (1), encoded in the causal structure, can be translated to linear constraints on the components of such entropy vectors.

The mapping to entropy vectors is particularly beneficial, as the set of entropy vectors that are compatible with a causal structure form a convex cone, which enables the application of linear programming tools. We use entropy vectors to derive the following results.

Improvement on the current entropic techniques [18]. We find that for many causal structures so called non-Shannon entropic inequalities [30] allow us to derive new constraints on the sets of possible entropy vectors in the classical case. Hence, they lead to tighter approximations of the set of realisable entropy vectors, which enables a sharper distinction of different causal structures.

We have for instance derived several (infinite) families of inequalities for the triangle scenario (cf. Figure 1(a)). These allow us to prove that previous approximations of this entropic cone were not tight (contrary to what has been claimed in the literature [14, 16]) and to improve on its distinction from other causal structures such as the one in Figure 1(b). For the triangle causal structure this is rather surprising, as for three variable distributions the entropic cones are tight; only the joint distribution with the three unobserved variables that we marginalise over allows for the application of additional non-Shannon constraints.

Regarding the quantum case, we have shown that for causal structures that involve up to five random variables there is never a distinction between classical and quantum entropic cones in this approach.[1] The same could not be established for causal structures that

[1]Note that this does neither contradict Braunstein and Caves' results for the Bell scenario [31], nor its generalisations [11, 13, 26]. These works all take conditioning the involved variables on particular

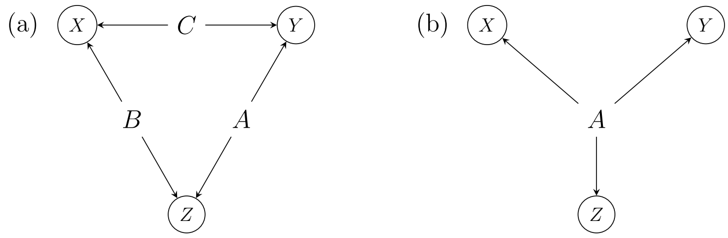

Fig. 1: Three-party correlations. (a) Triangle causal structure: Three observed random variables X, Y and Z have pairwise common causes A, B and C. In the classical case these common causes are random variables while in the quantum case these are replaced by quantum systems ρ_{A}, ρ_{B} and ρ_{C}. (b) Causal structure that allows for any three-party correlations (in the classical and quantum case).

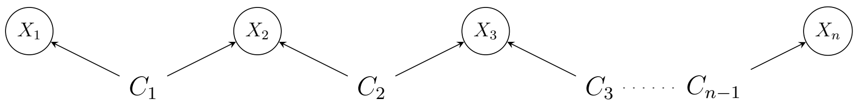

Fig. 2: A family of causal structures including the bipartite Bell scenario. There are n observed variables, $X_1, X_2, \ldots, X_n$, and $n-1$ unobserved ones, $C_1, C_2, \ldots, C_{n-1}$. The latter correspond to random variables in the classical case and to quantum systems if the causal structure is quantum.

involve six random variables (such as the triangle). Instead, we found that for all six variable causal structures where quantum correlations may allow for a larger set of joint distributions than classical hidden variables (all listed in [15]), non-Shannon inequalities play an important role for their entropic characterisation. A behaviour which is likely to extend to most causal structures that involve more nodes.

Whether these improved entropic characterisations are also valid in the quantum case remains an open problem whose resolution would have implications for the discrimination of classical and quantum cause and for the problem of whether there exist novel inequalities for the von Neumann entropy beyond strong subadditivity.

Problems for the distinction between classical and quantum cause [19]. We consider the question of whether entropy vectors can lead to useful certificates of non-classicality. We have established that for a family of causal structures that include the bipartite Bell scenario they do not, in spite of the existence of non-classical correlations in all of them (cf. Figure 2). The same is true for the bilocality causal structure, which is important in the context of entanglement swapping and finds technological application in quantum repeaters [32, 33].

Due to these stark limitations we analyse methods extending the entropy vector approach [11, 13, 26, 31], which allow for the derivation of entropic certificates for non-classicality in these and other causal structures. We find that, in general, these methods also benefit from the application of non-Shannon inequalities.

values of certain observed input nodes into account. They effectively consider modified causal structures and different types of marginals. Such methods are, however, not generally applicable, e.g. the triangle scenario does not fulfil the conditions that allow for the application of these approaches.

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