Irreducible noncontextuality inequalities from the Kochen-Specker theorem

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Recent work [1] has shown how operational noncontextuality inequalities robust to noise can be obtained from proofs of the Kochen-Specker theorem [2]. In contrast to traditional approaches, this operational approach [3] does not assume that measurement outcomes are fixed deterministically by the ontic state of the system in an underlying ontological model, nor does it employ factorizability à la Bell's theorem.[1] The result of Ref. [1] relied on an explicit numerical enumeration of all the extremal points of the polytope of (measurement) noncontextual assignments of probabilities to the KS-uncolourable hypergraph. Any KS set of quantum projectors, together with the corresponding eigenstates, satisfying the orthogonality relations depicted by a KS-uncolourable hypergraph violates these inequalities. While this approach works, in principle, for any KS construction, obtaining the upper bound on the inequality gets more computationally demanding as the size of the KS-uncolourable hypergraph grows.

In the present work [5], I develop an analytic approach to obtaining noncontextuality inequalities from proofs of the Kochen-Specker theorem. This analytic approach relies on constraints arising directly from the structure of the hypergraph without necessarily enumerating all the extremal probabilistic models on it. This cleanly identifies operational quantities that one can expect to be constrained (and why) by the assumption of noncontextuality instead of having to guess these quantities or obtaining them from brute-force numerical methods, such as Fourier-Motzkin elimination [6, 7], without any guiding principles to identify them. I obtain noncontextuality inequalities robust to noise for a whole family of KS-uncolourable hypergraphs (of arbitrarily large size), including many known KS constructions, using this method.

This allows identification of a minimal set of independent noncontextuality inequalities for any KS-uncolourable hypergraph given the operational equivalences of the type assumed in Ref. [1]. These inequalities are seen to underlie the coarse-grained inequalities of the type in Ref. [1], a fact that was first realized for this case in Ref. [6] by performing a Fourier-Motzkin elimination. Along the way to these noncontextuality inequalities, a parameterization of contextuality

[1] Factorizability is justified by the assumption of local causality in the case of Bell scenarios. No such justification is available in the case of noncontextuality experiments because of the generic lack of spacelike separation when prepare-and-measure experiments are done in a single laboratory [4]. In Bell scenarios, after the preparation and distribution of quantum systems to the parties, the measurements are done in spacelike separated laboratories: this allows one to invoke the assumption of local causality and thus justify factorizability.

scenarios to obtain conditions for their KS-uncolourability is defined. Also, I present a particularly simple way to generate a family of KS-uncolorable hypergraphs by defining a map, 2Reg(.), that takes any graph to a corresponding (2-regular) hypergraph. Some known examples of Kochen-Specker sets exhibit orthogonality relations from this family of KS-uncolourable hypergraphs. Operational noncontextuality inequalities for (2-regular) KS-uncolourable hypergraphs are then obtained using properties of the (simpler) underlying graphs, namely, those from which they can be obtained via 2Reg(.).[2]

In summary, the present work [5] builds on previous work in the following ways:

1. The numerical approach to obtaining the upper bound on the noncontextuality inequality of Ref. [1] is replaced by an analytical approach based on a deeper understanding of the structure of KS-uncolourable hypergraphs.

2. The fine-grained inequalities seen to be obtained from Fourier-Motzkin elimination [6, 7] can be motivated from different principles that do not need quantifier elimination to obtain the inequalities. We illustrate this for the case of a family of 2-regular KS-uncolourable hypergraphs.

3. It paves the way to understanding these noncontextuality inequalities in terms of hitherto unappreciated graph-theoretic properties associated with KS-uncolourable hypergraphs: in particular, the presence of minimally indeterministic sets of contexts, or MISCs, in these hypergraphs [5].

References

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[2] The preliminary draft at [5] restricts attention to these scenarios but I intend to supplement these examples with those not belonging to this family of scenarios. I'll also add discussion of some other connections that I haven't had a chance to fully develop yet but which should be done before QPL.

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