Spectral Presheaves, Kochen–Specker Contextuality, and Quantale–Valued Relations

Kevin Dunne

University of Strathclyde, Glasgow, Scotland.
`kevin.dunne@strath.ac.uk`

In the topos approach to quantum theory of Doering and Isham the Kochen–Specker Theorem, which asserts the contextual nature of quantum theory, can be reformulated in terms of the global sections of a presheaf characterised by the Gelfand spectrum of a commutative C^*–algebra. In previous work we showed how this topos perspective can be generalised to a class of categories typically studied within the monoidal approach to quantum theory of Abramsky and Coecke, and in particular how one can generalise the Gelfand spectrum. Here we study the Gelfand spectrum presheaf for categories of quantale–valued relations, and by considering its global sections we give a non–contextuality result for these categories. We also show that the Gelfand spectrum comes equipped with a topology which has a natural interpretation when thinking of these structures as representing physical theories.

1 Introduction

The present work is part of an ongoing project [10, 11] to marry conceptually the monoidal approach to quantum theory initiated by Abramsky and Coecke [3], and the topos approach to quantum theory initiated by Butterfield, Doering, and Isham [9, 17]. Both of these approaches to quantum theory are algebraic, in that they seek to represent some aspect of physical reality with algebraic structures. By taking the concept of a "physical observable" as a fixed point of reference we cast the difference between these approaches as internal vs. external algebraic perspectives; that is, algebras *internal* to a monoidal category $\mathscr{A}$ vs. representations of algebras on $\mathscr{A}$, a construction *external* to $\mathscr{A}$. The topos approach to quantum theory considers representations of commutative algebraic structures (for example C^*–algebras, or von Neumann algebras [8]) on **Hilb**. What makes the study of such representations on **Hilb** meaningful is the rich algebraic structure on the sets $\mathrm{Hom}(H,H)$, which are C^*–algebras. In [10] we showed that the categories considered in the monoidal approach have a similarly rich algebraic structure on their sets of endomorphisms $\mathrm{Hom}(A,A)$, thus allowing one to take the "external perspective" for any such $\mathscr{A}$, and not just **Hilb**.

There are various incarnations of the topos approach to quantum theory, here we follow a construction introduced in [9], which is developed in [12]. For a fixed Hilbert space H one takes **Hilb-Alg**(H) to be the poset of commutative C^*–subalgebras of $\mathrm{Hom}(H,H)$ considered as a category, and **Hilb-Alg**$_{\mathrm{vN}}(H)$ its subcategory whose objects are the commutative von Neumann C^*–subalgebras of $\mathrm{Hom}(H,H)$. We will briefly discuss a physical interpretation for this definition. Physical experiments have made clear that quantum mechanical systems are faithfully represented by non–commutative C^*–algebras of the form $\mathrm{Hom}(H,H)$. What nature does not make clear however is how to interpret this algebraic structure. According to Bohr's interpretation of quantum theory [5], although physical reality is by its nature quantum, as classical beings conducting experiments in our labs we only have access to the "classical parts" of a quantum system. Much of classical physics can be reduced to the study of commutative algebras; this approach is carefully constructed and motivated in [23] where the following picture is given:

Physics lab	$\rightarrow$	Commutative unital
		$\mathbb{R}$–algebra A
Measuring device	$\rightarrow$	Element of the algebra A
State of the observed	$\rightarrow$	Homomorphism of unital
physical system		$\mathbb{R}$–algebras $h : A \rightarrow \mathbb{R}$
Output of the	$\rightarrow$	Value of this function $h(a)$,
measuring device		$a \in A$

Figure 1: Algebraic interpretation of classical physics

In [23] the author stresses that the choice of ground ring is somewhat unimportant to this construction and interpretation, however $\mathbb{R}$ is a reasonable choice given that in classical physics most of the quantities we want to measure, length, energy, time, etc., can be represented by real numbers. In quantum mechanics one traditionally takes scalar values in $\mathbb{C}$, but one can take any ring, or, as we will see, a semiring in its place and the physical interpretation of Figure 1. remains valid.

According to Bohr's interpretation of quantum theory, a quantum system represented by a non–commutative algebra $\mathrm{Hom}(H, H)$, can only be understood in terms of of its classical components, that is, the commutative subalgebras of $\mathrm{Hom}(H, H)$; in particular, we can only make observations on the classical subsystems. Hence the category $\mathbf{Hilb\text{-}Alg_{vN}}(H)$ is a collection of observable subsystems of a physical system, and we consider all of these subsystems at one by considering the topos of presheaves $[\mathbf{Hilb\text{-}Alg_{vN}}(H)^{\mathrm{op}}, \mathbf{Set}]$. One presheaf of particular significance is the presheaf which characterises the *Gelfand spectrum*. Recall the Gelfand spectrum of a commutative C^*–algebra $\mathbf{A}$ is characterised by the set $\mathrm{Spec_G}(\mathbf{A}) = \{\, \rho : \mathbf{A} \rightarrow \mathbb{C} \mid \rho$ a C^*–algebra homomorphism $\}$ of *characters* which defines a functor

$$\mathbf{Hilb\text{-}Alg_{vN}}(H)^{\mathrm{op}} \xrightarrow{\ \mathrm{Spec_G}\ } \mathbf{Set}$$

with the action on morphisms given by restriction. By Figure 1. we interpret this functor as assigning to each classical subsystem its set of possible states.

Remark 1.1. The *prime spectrum* $\mathrm{Spec_P}(\mathbf{A})$ of a commutative C^*–algebra $\mathbf{A}$ is defined to be the set of prime ideals of $\mathbf{A}$, and is naturally isomorphic to the Gelfand spectrum. The correspondence comes from the fact that an ideal $J \subset \mathbf{A}$ is prime if and only if it is the kernel of a character $\rho : \mathbf{A} \rightarrow \mathbb{C}$. The prime spectrum is also equivalent to the *maximal spectrum*, taken to be the collection of maximal ideals.

In a presheaf category one can generalise the notion of elements of a set by considering the morphisms out of the terminal object. The terminal object $T : \mathscr{C}^{\mathrm{op}} \rightarrow \mathbf{Set}$ in a presheaf category sends all objects to the singleton set $\{*\}$ and all morphisms to the identity $\mathrm{id} : \{*\} \rightarrow \{*\}$. A *global section* (or *global element*) of a presheaf $F : \mathscr{C}^{\mathrm{op}} \rightarrow \mathbf{Set}$ is a natural transformation $\chi : T \rightarrow F$.

The Kochen–Specker Theorem [21] asserts the *contextual* nature of quantum theory. The principle of *non–contextuality* is that the outcome of a measurement should not depend on the *context* in which that measurement is performed, that is, it should not depend on which other measurements are made simultaneously. Classical physics is typically formulated as non–contextual [18, Chap. 4]. The Kochen–Specker Theorem states that it is a feature of quantum theory that one can find collections of measurements for which the outcomes are context dependent. For a mathematical treatment of this theorem see in [18, Chap. 9]. The following theorem was first shown in [9] but here we present it as in [12, Corollary 9.1].

Theorem 1.2. *The Kochen-Specker Theorem is equivalent to the statement that for a Hilbert space H with* $\dim(H) \geq 3$, *the presheaf* $\mathrm{Spec}_\mathrm{G} : \textbf{Hilb-Alg}_\mathrm{vN}(H)^{\mathrm{op}} \to \textbf{Set}$ *has no global sections.*

The monoidal approach to quantum theory of Abramsky and Coecke [3] is an entirely separate approach to quantum theory using very different mathematical structures. This approach begins with identifying the essential properties of the category **Hilb** which one needs to formulate concepts from quantum theory.

Definition 1.3. A †–*category* consists of a category $\mathscr{A}$ together with an identity on objects functor $\dagger : \mathscr{A}^{\mathrm{op}} \to \mathscr{A}$ satisfying $\dagger \circ \dagger = \mathrm{id}_\mathscr{A}$. A †–*symmetric monoidal category* consists of a symmetric monoidal category $(\mathscr{A}, \otimes, I)$ which is a †–category such that † is a strict monoidal functor and all of the symmetric monoidal structure isomorphisms satisfy $\lambda^{-1} = \lambda^\dagger$.

Definition 1.4. A category $\mathscr{A}$ is said to have *finite biproducts* if it has a zero object 0, and if for all objects X_1 and X_2 there exists an object $X_1 \oplus X_2$ which is both the coproduct and the product of X_1 and X_2. If $\mathscr{A}$ is a †–category and has finite biproducts such that the coprojections $\kappa_i : X_i \to X_1 \oplus X_2$ and projections $\pi_i : X_1 \oplus X_2 \to X_i$ are related by $\kappa_i^\dagger = \pi_i$, then we say $\mathscr{A}$ has *finite †–biproducts*.

In a category with a zero object 0, for every pair of objects X and Y we call the unique map $X \to 0 \to Y$ the *zero–morphism*, which we denote $0_{X,Y} : X \to Y$, or simply $0 : X \to Y$.

For a category with finite biproducts each hom-set $\mathrm{Hom}(X, Y)$ is equipped with a commutative monoid operation [22, Lemma 18.3] called *biproduct convolution*, where for $f, g : X \to Y$, we define $f + g : X \to Y$ by the composition

$$X \xrightarrow{\ \Delta\ } X \oplus X \xrightarrow{\ f \oplus g\ } Y \oplus Y \xrightarrow{\ \nabla\ } Y$$

where the monoid unit is given by the zero–morphism $0_{X,Y} : X \to Y$.

Categories with finite †–biproducts admit a matrix calculus [22, Chap. I. Sect. 17.] characterised as follows. For $X = \bigoplus_{j=1}^{n} X_j$ and $Y = \bigoplus_{i=1}^{m} Y_i$ a morphism $f : X \to Y$ is determined completely by the morphisms $f_{i,j} : X_i \to Y_j$, and morphism composition is given by matrix multiplication. If f has matrix representation $f_{i,j}$ then $f^\dagger$ has matrix representation $f_{j,i}^\dagger$.

The category **Hilb** is the archetypal example of a category with these properties. A notion of "observable" in quantum theory can be axiomatised in terms of the monoidal structure of the category of Hilbert spaces by Frobenius algebras [7] or H^*–algebras [4], and hence any †–symmetric monoidal category serves as a categorical model for a toy theory of observables. For example, Spekkens Toy Theory [26] which is a toy physical theory exhibiting some quantum–like properties but which is given by a local hidden variable model. This theory can be modelled in the category of sets and relations **Rel** using Frobenius algebras to represent observables [6].

The monoidal approach provides general framework in which a broad class of physical theories can be compared in a high–level but completely rigorous way. This is useful for exploring interdependencies of quantum or quantum–like phenomena, for example the many notions of non–locality and contextuality. In particular, in [14] an abstract notion of Mermin–locality is formulated in the language of Frobenius algebras, and the category of finite sets and relations **FRel** is shown to be Mermin–local.

In this work we present a completely abstract notion of Kochen–Specker contextuality and we show that categories of quantale–valued relations do not exhibit this form of contextuality. This is done using abstract Gelfand spectrum introduced in [10]. In order to prove this non–contextuality result we define the *prime spectrum* which we relate to the physical interpretation of Figure 1. by examining the topological structure which these spectra carry.

2   The Spectrum and Kochen–Specker Contextuality

In this section we review a construction introduced in [10], and we introduce an abstract definition of Kochen–Specker contextuality. This is done using the language of semirings, semimodules [15] and semialgebras.

Definition 2.1. A *semiring* $(R, \cdot, 1, +, 0)$ consists of a set R equipped with a commutative monoid operation $+ : R \times R \to R$ with unit $0 \in R$, and a monoid operation $\cdot : R \times R \to R$, with unit $1 \in R$, such that $\cdot$ distributes over $+$ and such that $s \cdot 0 = 0 \cdot s = 0$ for all $s \in R$.

    A semiring is called *commutative* if $\cdot$ is commutative. A $*$-*semiring*, or *involutive semiring* is one equipped with an operation $* : R \to R$ which is an involution, a homomorphism for $(R, +, 0)$, and satisfies $(s \cdot t)^* = t^* \cdot s^*$ and $1^* = 1$.

    As the notation suggests we will refer to the monoid operations of a semiring as *addition* and *multiplication* respectively. We say that a semiring R is *zero–divisor free (ZDF)* if for all $s, t \in R$ we have $s \cdot t = 0$ implies $s = 0$ or $t = 0$. Many concepts associated with rings can be lifted directly to the level of semirings in the obvious way, for example homomorphisms, kernels and direct sums. However, some concepts have to be treated with care when generalising to semirings, for example *ideals*.

Definition 2.2. Let R be a commutative semiring. A subset $J \subset R$ is called an *ideal* if it contains 0, is closed under addition, and such that for all $s \in R$ and $a \in J$, $as \in J$. An ideal is called *prime* if $st \in J$ implies $s \in J$ or $t \in J$. A k–*ideal* is an ideal J such that if $a \in J$ and $a + b \in J$ then $b \in J$. A k^*–ideal of a $*$–semiring is a k–ideal closed under involutions.

    The k–ideals are to a semirings what normal subgroups are to a groups; they are the ideals which one can quotient by. For any ring considered as a semiring the ideals and k–ideals coincide.

Definition 2.3. Let $(R, \cdot, 1, +, 0)$ be a commutative semiring, an R–*semimodule* consists of a commutative monoid $+_M : M \times M \to M$, with unit 0_M, together with a *scalar multiplication* $\bullet : R \times M \to M$ such that for all $r, s \in R$ and $m, n \in M$:

1. $s \bullet (m +_M n) = s \bullet m +_M s \bullet n$;
2. $(r \cdot s) \bullet m = r \bullet (s \bullet m)$;
3. $(r + s) \bullet m = (r \bullet m) +_M (s \bullet m)$;
4. $0 \bullet m = s \bullet 0_M = 0_M$;
5. $1 \bullet m = m$.

Definition 2.4. An R–*semialgebra* $(M, \cdot_M, 1_M, +_M, 0_M)$ consists of an R-semimodule $(M, +_M, 0_M)$ equipped with a monoid operation $\cdot_M : M \times M \to M$, with unit 1_M, such that $(M, \cdot_M, 1_M, +_M, 0_M)$ forms a semiring, and where scalar multiplication obeys $s \bullet (m \cdot_M n) = (s \bullet m) \cdot_M n = m \cdot_M (s \bullet n)$. An R–semialgebra is called *commutative* if $\cdot_M$ is commutative.

    The ideals and k–ideals of a semialgebra are defined in the obvious way. Notice that every semiring R is an R-semialgebra, where the scalar multiplication by R is taken to be the usual multiplication in R. Non–zero elements s, t of a semialgebra are *orthogonal* if $s \cdot t = 0$. A *subunital idempotent* in a semialgebra is an idempotent element p such that there is an orthogonal idempotent q where $p + q = 1_M$. A *primitive subunital idempotent* is a subununital idempotent p such that there exists no non–trivial subunital idempotents s and t with $s + t = p$.

Definition 2.5. Let R be a $*$–semiring. An R^*–*semialgebra* consists of an R–semialgebra $(M, \cdot_M, 1_M, +_M, 0_M)$, such that M and R have compatible involutions, that is, one that satisfies $(s \bullet m)^* = s^* \bullet m^*$.

A *unital subsemialgebra* $i : N \hookrightarrow M$ of M is a subset N containing 0_M and 1_M closed under all the algebraic operations. A *subsemialgebra* $N \subset M$ is a subset N containing 0_M which is closed under multiplication and which is a semialgebra in its own right, though may have a different unit from M. A *(unital) *–subsemialgebra* of a *–semialgebra is a (unital) subsemialgebra closed under taking involutions.

An *R*–semialgebra is said to be *indecomposable* if it cannot be expressed as a non–trivial direct sum of *R*–semialgebras. An *R*–semialgebra is *completely decomposable* if it is isomorphic to the direct sum of its indecomposable subsemialgebras.

The following two results are shown in detail in [10].

Theorem 2.6. *For a locally small †–symmetric monoidal category $(\mathscr{A}, \otimes, I)$ with finite †–biproducts the set $S = \mathrm{Hom}(I, I)$ is a commutative *–semiring.*

Biproduct convolution gives us the additive operation on S while morphism composition gives us the multiplicative operation, and the functor † provides the involution. It is shown in [20] that this multiplicative operation is commutative.

Theorem 2.7. *Let $(\mathscr{A}, \otimes, I)$ be a locally small †–symmetric monoidal category and let $S = \mathrm{Hom}(I, I)$. For any pair of objects the set $\mathrm{Hom}(X, Y)$ is an S–semimodule, and $\mathrm{Hom}(X, X)$ is a S^*–semialgebra.*

Addition on the set $\mathrm{Hom}(X, Y)$ is given by biproduct convolution. For a morphism $f : X \to Y$ scalar multiplication $s \bullet f$ for $s : I \to I$ is defined

$$X \xrightarrow{\ \sim\ } X \otimes I \xrightarrow{\ f \otimes s\ } Y \otimes I \xrightarrow{\ \sim\ } Y$$

which gives a semimodule structure [16]. For $\mathrm{Hom}(X, X)$ multiplication is given by morphism composition and the functor † provides the involution.

Definition 2.8. For $(\mathscr{A}, \otimes, I)$ a locally small †–symmetric monoidal category and X and object, we define the category $\mathscr{A}\text{-}\mathbf{Alg}(X)$ to be the category with objects commutative unital S^*–subsemialgebras of $\mathrm{Hom}(X, X)$, and arrows inclusion of unital subsemialgebras.

The for any subset of $B \subset \mathrm{Hom}(X, X)$ the set $B' = \{\ f : X \to X \mid f \circ g = g \circ f \text{ for all } g \in B\ \}$ is called the *commutant* of B [8, Sect. 12]. We define the full subcategory of *von Neumann S^*–subsemialgebras*

$$\mathscr{A}\text{-}\mathbf{Alg}_{\mathrm{vN}}(X) \lhook\joinrel\longrightarrow \mathscr{A}\text{-}\mathbf{Alg}(X)$$

to have objects those S^*–subsemialgebras $\mathbf{A}$ which satisfy $\mathbf{A} = \mathbf{A}''$.

Example 2.9. If we take $(\mathscr{A}, \otimes, I)$ to be $(\mathbf{Hilb}, \otimes, I)$ then the category $\mathbf{Hilb}\text{-}\mathbf{Alg}_{\mathrm{vN}}(H)$ is precisely the category considered in the topos approach [9, 12].

Remark 2.10. In [11] we showed that any special commutative unital Frobenius algebra, and any (possibly non–unital) commutative H^*–algebra $\mu : A \otimes A \to A$ in $\mathscr{A}$ generates an object $\mathbf{A}$ in $\mathscr{A}\text{-}\mathbf{Alg}_{\mathrm{vN}}(A)$. Furthermore, there is a natural correspondence between the set–like elements of the internal algebra and the Gelfand spectrum of the semialgebra it generates. Hence the notion observable in the monoidal approach naturally lifts to the notion of observable in the generalised topos approach.

We can generalise the spectrum of a commutative C^*–algebra to an S^*–semialgebra [10].

Definition 2.11. Let $(\mathscr{A}, \otimes, I)$ be a locally small †–symmetric monoidal category with finite †–biproducts, and X an object. The *Gelfand spectrum* for X is the presheaf

$$\mathscr{A}\text{-}\mathbf{Alg}_{\mathrm{vN}}(X)^{\mathrm{op}} \xrightarrow{\ \mathrm{Spec}_{\mathrm{G}}\ } \mathbf{Set}$$

defined on objects $\mathrm{Spec}_{\mathrm{G}}(\mathbf{A}) = \{\ \rho : \mathbf{A} \to S \mid \rho \text{ an } S^*\text{–semialgebra homomorphism}\ \}$ to be the set of *characters* while the action on morphism is defined in the obvious way by restriction (precomposition).

Definition 2.12. Let $(\mathscr{A}, \otimes, I)$ be a locally small †–symmetric monoidal category with finite †–biproducts, and X an object. The *prime spectrum* for X is the presheaf

$$\mathscr{A}\text{-}\mathbf{Alg}_{vN}(X)^{op} \xrightarrow{\text{Spec}_{P}} \mathbf{Set}$$

defined on objects $\text{Spec}_P(\mathbf{A}) = \{\, J \subset \mathbf{A} \mid J \text{ a prime } k^*\text{–ideal} \,\}$ while for $i : \mathbf{A} \hookrightarrow \mathbf{B}$ the action on morphisms is given by $\tilde{i} : \text{Spec}_P(\mathbf{B}) \to \text{Spec}_P(\mathbf{A})$ is defined $\tilde{i}(K) = \{\, x \in \mathbf{A} \mid i(x) \in K \,\}$.

To see that $\tilde{i}(K)$ is a prime k^*–ideal one can see the proof of a similar statement [15, Proposition 6.13].

Remark 2.13. One can also define a functor which assigns to each $\mathbf{A}$ the collection of all prime ideals, not just the prime k^*–ideals [15, Chap. 6], although for the purposes of this work k^*–ideals are a more natural choice. One can define the maximal spectrum for an arbitrary semialgebra or semiring, although this fails to be functorial in general, [25, Chap 2. Sect. 5].

We have already discussed that for **Hilb** the prime spectrum and Gelfand spectrum coincide. In [10] we showed that the same is true for the category of sets and relations **Rel**, although we will see in Example 3.12 that this is not the case in general.

The Gelfand spectrum presheaf formulation of the Kochen–Specker Theorem justifies the following definition.

Definition 2.14. Let $\mathscr{A}$ be a locally small †–symmetric monoidal category with finite †–biproducts. An object X in $\mathscr{A}$ is said to be *Kochen–Specker contextual* if the presheaf Spec_G on $\mathscr{A}\text{-}\mathbf{Alg}_{vN}(X)$ has no global sections. We say X is *Kochen–Specker non–contextual* if Spec_G does admit a global section.

Such a global section, if it exists, will pick out an element $\chi_{\mathbf{A}} : \{*\} \to \text{Spec}_G(\mathbf{A})$ from each spectrum, i.e. according to Figure 1. it would specify a state from each classical subsystem $\mathbf{A}$. Naturality ensures that these choices of states are consistent with measurement outcomes irrespective of which subsystem – that is, which "context" – the measurement appears in.

There are more general formulations of contextuality and non–locality using the language of sheaves and presheaves [2, 1]. Future work will show how the categories $\mathscr{A}\text{-}\mathbf{Alg}_{vN}(X)$ naturally generate *empirical models* which can be examined within the framework of [1], and how the contextual nature of those empirical models is related to the existence of global sections for the corresponding Gelfand spectrum. This connection, together with Remark 2.10 will give us a means of applying the techniques of [1], for example sheaf cohomology, to the Frobenius algebras in an arbitrary $\mathscr{A}$.

3   Quantale–Valued Relations

We now turn our attention to a class of categories for which we will prove a non–contextuality result, namely quantale–valued relations over a fixed quantale Q. A standard reference for quantales is [24].

Definition 3.1. A *quantale* $(Q, \bigvee, \cdot, 1_Q)$ is a complete join–semilattice $(Q, \bigvee)$ equipped with a monoid operation $\cdot : Q \times Q \to Q$ with unit 1_Q such that for any $x \in Q$ and $P \subset Q$

$$x \cdot \left(\bigvee_{y \in P} y\right) = \bigvee_{y \in P} (x \cdot y) \qquad \text{and} \qquad \left(\bigvee_{y \in P} y\right) \cdot x = \bigvee_{y \in P} (y \cdot x)$$

An *involutive quantale* in one equipped with an involution map $* : Q \to Q$ which is a semilattice homomorphism which is an involution $(x^*)^* = x$ satisfying $(x \cdot y)^* = y^* \cdot x^*$ and $1_Q^* = 1_Q$. A *commutative quantale* is one for which the monoid operation is commutative. A *subquantale* is a subset of Q closed under all joins and the monoid operation and containing 1_Q.

We are primarily interested in involutive commutative quantales, but note every commutative quantale can be equipped with the trivial involution. A quantale has a least element $\perp$, defined to be the join of the empty set, and this is an absorbing element, i.e. for all $x \in Q$ we have $x \cdot \perp = \perp$. We assume all quantales are non–trivial, that is, $\perp \neq \top$, where $\top = \bigvee_{x \in Q} x$.

Remark 3.2. An involutive quantale Q is a $*$–semiring with addition given by the join and multiplication given by the monoid operation. The bottom element $\perp$ is the zero element of the semiring and will hence be denoted 0. We say a quantale is *zero–divisor free* if it is zero–divisor free as a semiring.

Definition 3.3. For a commutative involutive quantale Q, the category of *quantale–valued relations* $\mathbf{Rel}_Q$ has sets as objects and morphisms $f : X \to Y$ consist of functions $f : X \times Y \to Q$. For $f : X \to Y$ and $g : Y \to Z$ composition is defined where $g \circ f : X \times Z \to Q$ by $g \circ f(x,z) = \bigvee_{y \in Y} f(x,y) \cdot g(y,z)$. We say that a morphism $f : X \to Y$ in $\mathbf{Rel}_Q$ *relates* $x \in X$ to $y \in Y$ if $f(x,y) \neq 0$.

The category $\mathbf{Rel}_Q$ is a †–symmetric monoidal category with †–biproducts. The monoidal product is given by the cartesian product, with unit the one element set, the biproduct is given by disjoint union, and the dagger is given by reordering and pointwise application of the involution $f^\dagger(y,x) = f(x,y)^*$.

Example 3.4. Any complete Heyting algebra or Boolean algebra is a quantale. In particular the two–element Boolean algebra $\mathbf{2} = \{0, 1\}$, where the corresponding category $\mathbf{Rel_2}$ is the usual the category of sets and relations $\mathbf{Rel}$. The intervals $[0, 1]$ and $[0, \infty]$ are quantales when equipped with the usual multiplication, and where $\bigvee S = \sup S$.

We now turn our attention to the category $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{vN}(X)$ for a set X. Clearly the scalars $\mathrm{Hom}(I, I) \cong Q$, and for each set X the Q–semialgebra $\mathrm{Hom}(X, X)$ is a quantale, with the join given pointwise and multiplication given by morphism composition.

Lemma 3.5. *Each $\mathbf{A}$ in $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{vN}(X)$ is a subquantale of* $\mathrm{Hom}(X, X)$.

Proof. By definition $\mathbf{A}$ is a subsemiring, we need to show that $\mathbf{A}$ is closed under arbitrary joins. Let $B \subset \mathbf{A}$ be any subset, we need to show that $\bigvee_{x \in B} x \in \mathbf{A}$. Let $g \in \mathbf{A}'$ then for all $x \in B$ we have $g \cdot x = x \cdot g$. So we have $g \cdot \left(\bigvee_{x \in B} x \right) = \bigvee_{x \in B} (g \cdot x) = \bigvee_{x \in B} (x \cdot g) = \left(\bigvee_{x \in B} x \right) \cdot g$ and hence $\bigvee_{x \in B} x \in \mathbf{A}''$, and since $\mathbf{A}$ is von Neumann $\bigvee_{x \in B} x \in \mathbf{A}$, as required. $\qquad\square$

We now give an important structure theorem for these semialgebras.

Theorem 3.6. *Let $(Q, \leq, \bigvee, \perp, \cdot, 1_Q)$ be a commutative ZDF quantale and let $\mathbf{A} \in \mathbf{Rel}_Q\text{-}\mathbf{Alg}_{vN}(X)$. There are primitive subunital idempotents $\{e_i\}$ such that $\mathbf{A} = \prod_i e_i \mathbf{A}$, a direct product of S^*–semialgebras.*

Proof. Let $f : X \to X$ be a Q–relation. Let $\mathrm{supp}(f) \subset X$, the *support* of f be the set of elements x such that there exits $y \in X$ such that f relates x to y. Let $\mathrm{cosupp}(f) \subset X$, the *cosupport* of f be the set of elements x such that there exists $y \in X$ such that f relates y to x. First, we claim that for Q–relations satisfying $f \circ f^\dagger = f^\dagger \circ f$ we have $\mathrm{supp}(f) = \mathrm{cosupp}(f)$. Suppose $x \in \mathrm{supp}(f)$ then if Q is ZDF then $f^\dagger \circ f$ relates x to itself. However if $x \notin \mathrm{cosupp}(f)$ then clearly $f \circ f^\dagger$ cannot relate x to any other element and hence $x \in \mathrm{supp}(f)$ iff $x \in \mathrm{cosupp}(f)$. So $X = \mathrm{supp}(f) \sqcup \overline{\mathrm{supp}(f)}$ and f has a corresponding matrix representation $f = \left(\begin{smallmatrix} f_1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$. For each $f \in \mathbf{A}$ let $f_{\mathrm{supp}} = \left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & 0 \end{smallmatrix} \right)$ be the relation which is the identity on the support of f and zero otherwise.

Let $g = \left(\begin{smallmatrix} g_1 & g_2 \\ g_3 & g_4 \end{smallmatrix} \right) \in \mathbf{A}'$ then in particular $g \circ f = f \circ g$ and hence $\left(\begin{smallmatrix} g_1 f_1 & 0 \\ g_3 f_1 & 0 \end{smallmatrix} \right) = \left(\begin{smallmatrix} f_1 g_1 & f_2 g_2 \\ 0 & 0 \end{smallmatrix} \right)$ and so if Q is ZDF then $g_2 = 0$ and $g_3 = 0$, and hence $g = \left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_4 \end{smallmatrix} \right)$. Then clearly $\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_4 \end{smallmatrix} \right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)\left(\begin{smallmatrix} g_1 & 0 \\ 0 & g_4 \end{smallmatrix} \right)$, i.e.

$f_{\text{supp}} \circ g = g \circ f_{\text{supp}}$, and hence we have shown that $f_{\text{supp}} \in \mathbf{A}''$, and hence by the assumption that $\mathbf{A}$ is von Neumann we have $f_{\text{supp}} \in \mathbf{A}$. By a similar argument $f_{\overline{\text{supp}}} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \text{id} \end{smallmatrix}\right)$ also belongs to $\mathbf{A}$.

Consider the collection of elements f_{supp} for all $f \in \mathbf{A}$. Each f_{supp} corresponds with a subset of X and hence this collection forms a Boolean subalgebra of $P(X)$, the powerset of X. By Lemma 3.5 $\mathbf{A}$ has all joins and hence this collection of subunital maps forms a complete Boolean subalgebra of $P(X)$ which by [13, Chap. 14, Theorem 8] is atomic. The atoms e_i of this Boolean algebra are the primitive subunital idempotents of $\mathbf{A}$, and $1_{\mathbf{A}} = \bigvee e_i$. For every element $f \in \mathbf{A}$ we have $f = \bigvee f \circ e_i$ for pairwise orthogonal subunital idempotents, and hence $\mathbf{A}$ is the direct product of the subalgebras $e_i\mathbf{A}$. $\square$

For a commutative ZDF quantale Q and for any set X Theorem 3.6 states that all semialgebras in $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\text{vN}}(X)$ are *completely decomposable*, that is, a direct sum of their indecomposable subalgebras. We call each $e_i\mathbf{A}$ for e_i a primitive subunital idempotent an *indecomposable component* of $\mathbf{A}$.

We now give a characterisation of the prime spectrum for semialgebras of quantale–valued relations.

Lemma 3.7. *For Q a commutative ZDF quantale and $\mathbf{A}$ an object in $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\text{vN}}(X)$, then $J \subset \mathbf{A}$ is a k^*–prime ideal iff if it the kernel of some S^*–semialgebra homomorphism $\gamma : \mathbf{A} \to \mathbf{2}$.*

Lemma 3.8. *For each semialgebra homomorphism $\gamma : \mathbf{A} \to \mathbf{2}$ there is exactly one primitive subunital idempotent e_a in $\mathbf{A}$ such that $\gamma(e_a) = 1$.*

Proof. First we show there is at most one primitive idempotent e_a such that $\gamma(e_a) = 1$. Suppose there is another e_b such that $\gamma(e_b) = 1$. Since e_a and e_b are orthogonal we have $\gamma(e_a)\gamma(e_b) \neq \gamma(e_a \circ e_b)$, and hence there is at most one e_a such that $\gamma(e_a) = 1$. Suppose however there are no primitive idempotents which map to 1. We still have $\gamma(1_{\mathbf{A}}) = \gamma(\bigvee e_i) = 1$. If there is only a finite number of primitive idempotents then we have $\gamma(\bigvee e_i) = \bigvee \gamma(e_i)$, a contradiction, and hence there is exactly one primitive idempotent satisfying $\gamma(e_a) = 1$. Suppose then that there are an infinite number of primitive idempotents. Partition the primitive idempotents into two infinite sets K and L. Then $\gamma(1_{\mathbf{A}}) = \gamma(\bigvee e_i) = \gamma(\bigvee_K e_k) + \gamma(\bigvee_L e_l) = 1$ but clearly $\gamma(\bigvee_K e_k)\gamma(\bigvee_L e_l) = 0$ and hence either $\gamma(\bigvee_K e_k) = 0$ or $\gamma(\bigvee_L e_l) = 0$. Suppose $\gamma(\bigvee_L e_l) = 0$ then by Lemma $\ker(\gamma)$ is a prime ideal, however there are elements e_{k_1} and e_{k_2} in K and therefore not in $\ker(\gamma)$ and hence we have $e_{k_1} \cdot e_{k_2} = 0$ contradicting the primeness of $\ker(\gamma)$, and hence for each semialgebra homomorphism $\gamma : \mathbf{A} \to \mathbf{2}$ there is exactly one primitive idempotent such that $\gamma(e_a) = 1$. $\square$

Theorem 3.9. *For Q a ZDF quantale and $\mathbf{A}$ in $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\text{vN}}(X)$ with decomposition $\mathbf{A} = \prod_i e_i\mathbf{A}$. For each indecomposable subalgebra $e_a\mathbf{A}$ the complement $\overline{e_a\mathbf{A}} = \prod_{i \neq a} e_i\mathbf{A}$ of $e_a\mathbf{A}$ is a prime ideal.*

Proof. This follows directly form Lemma 3.8, simply define the map $\gamma_a : \mathbf{A} \to \mathbf{2}$ with kernel $\overline{e_a\mathbf{A}}$, sending all other elements to 1. $\square$

Although we will see in Example 3.12 that the prime spectrum and the Gelfand spectrum for $\mathbf{Rel}_Q$ do not coincide in general, the following theorem shows that they are related.

Lemma 3.10. *For Q a commutative ZDF quantale there are natural transformations $\xi : \text{Spec}_{\text{G}} \to \text{Spec}_{\text{P}}$ and $\tau : \text{Spec}_{\text{P}} \to \text{Spec}_{\text{G}}$ such that $\xi \circ \tau \cong \text{id}$.*

Proof. For Q a quantale there is exactly one quantale homomorphism $! : \mathbf{2} \to Q$. For Q a ZDF quantale there is at least one homomorphism $w : Q \to \mathbf{2}$, which sends all non–zero elements to 1. Since Spec_{P} can be characterised by the collection of homomorphisms $\gamma : \mathbf{A} \to \mathbf{2}$ let $\tau(\gamma) = ! \circ \gamma$. Similarly for $\rho : \mathbf{A} \to Q$ define $\xi(\rho) = w \circ \rho$. Naturality is easy to check and clearly $w \circ ! \circ \gamma = \gamma$, as required. $\square$

In Sect. 5 we discuss a topological interpretation of this map $\xi_{\mathbf{A}} : \mathrm{Spec}_G(\mathbf{A}) \to \mathrm{Spec}_P(\mathbf{A})$, in particular how to relate the prime spectrum to the state space interpretation of the Gelfand spectrum of Figure 1.

The following theorem follows directly from Lemma 3.10.

Theorem 3.11. *For a ZDF quantale Q, the Gelfand spectrum for $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(X)$ has a global section if and only if the prime spectrum has a global section.*

Example 3.12. Let Q be the commutative involutive quantale $[0,1]$ with usual multiplication, trivial involution, and where $\bigvee S = \sup S$. Let X be a two element set and consider $\mathbf{A}$ the von Neumann Q–semialgebra $\mathbf{A} = \left\{ \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix} \right) \mid p, q \in Q \right\} \cong Q \oplus Q$ It is easy to see that there are four elements of $\mathrm{Spec}_P(\mathbf{A})$:

$$J_1 = \left\{ \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix} \right) \mid p \in Q \right\} \qquad J_2 = \left\{ \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix} \right) \mid p \in Q,\ q < 1 \right\}$$

$$K_1 = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & q \end{smallmatrix} \right) \mid q \in Q \right\} \qquad K_2 = \left\{ \left(\begin{smallmatrix} p & 0 \\ 0 & q \end{smallmatrix} \right) \mid q \in Q,\ p < 1 \right\}$$

There are three semialgebra homomorphisms from Q to itself: $u : Q \to Q$ defined $u(x) = 1$ for all $x \neq 0$; $d : Q \to Q$ defined $d(x) = 0$ for all $x < 1$; and the identity $\mathrm{id} : Q \to Q$. Hence there are six homomorphisms

$$\varphi_1 = \langle d, 0 \rangle : Q \oplus Q \to Q \qquad \varphi_2 = \langle u, 0 \rangle : Q \oplus Q \to Q \qquad \varphi_3 = \langle \mathrm{id}, 0 \rangle : Q \oplus Q \to Q$$

$$\theta_1 = \langle 0, d \rangle : Q \oplus Q \to Q \qquad \theta_2 = \langle 0, u \rangle : Q \oplus Q \to Q \qquad \theta_3 = \langle 0, \mathrm{id} \rangle : Q \oplus Q \to Q$$

corresponding to the six elements of $\mathrm{Spec}_G(\mathbf{A})$.

4   A Proof of Non–Contextuality

We now show that for a ZDF quantale Q every object X in $\mathbf{Rel}_Q$ is Kochen–Specker non–contextual. We do this by showing that picking an element from the underlying set X allows one to construct a global section of Spec_P. By Theorem 3.11 we can then conclude that Spec_G has global sections and thus every object in $\mathbf{Rel}_Q$ is Kochen–Specker non–contextual. We then show a partial converse to this result, that every global section for Spec_P in turn picks out an element from X.

Theorem 4.1. *For Q a commutative ZDF quantale, and X a set, each $x \in X$ determines a global section of the prime spectrum $\mathrm{Spec}_P : \mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(X)^{\mathrm{op}} \to \mathbf{Set}$.*

Proof. We show that each element $x \in X$ determines a global section. By Theorem 3.6 each semialgebra $\mathbf{A}$ in $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(X)$ has a decomposition $\prod_i e_i \mathbf{A}$ for subunital idempotents e_i. Note that x is in the support of exactly one of the primitive subunital idempotents, which we will denote e_x. By Theorem 3.9 $\overline{e_x \mathbf{A}}$ is a prime ideal. Let $\tilde{x}_{\mathbf{A}} : \mathbf{A} \to \mathbf{2}$ be the map corresponding to this prime ideal defined $\tilde{x}_{\mathbf{A}}(e_x) = 1$. The claim is that $\tilde{x}$ determines a natural transformation. We need to show that for each $\mathbf{A} \hookrightarrow \mathbf{B}$ that the restriction of $\tilde{x}_{\mathbf{B}}$ so $\mathbf{A}$ is equal to $\tilde{x}_{\mathbf{A}}$. Let $\mathbf{B} = \prod_j d_j \mathbf{B}$ with $\tilde{x}_{\mathbf{B}}(d_x) = 1$. Since e_x and d_x both relate x to itself we have $e_x \circ d_x \neq 0$. Clearly then $e_x \circ d_x$ is a non–zero element of the subsemialgebra $d_x \mathbf{B} \subset \mathbf{B}$ and hence $\tilde{x}(e_x \circ d_x) = 1$. This implies that $\tilde{x}_{\mathbf{B}}(e_x) = 1$ and therefore $\tilde{x}_{\mathbf{A}}(e_x) = \tilde{x}_{\mathbf{B}}(e_x)$. $\square$

Central to the proof of Theorem 4.1 is reducing the problem to consider the partitions of the underlying set X. The proof of the Kochen–Specker Theorem also reduces the problem to a consideration of the "partitions" on the Hilbert space H, that is, the orthonormal bases of H. At the heart of the difference between the contextuality results for **Hilb** and $\mathbf{Rel}_Q$ is that given an element of a set X we can pick a

component from every partition of X in a canonical way. However, for a Hilbert space if we choose a vector $|\psi\rangle \in H$ there is not a canonical way of picking an element from each orthonormal basis of H.

This non–contextuality result for $\mathbf{Rel}_Q$ is consistent with a theorem which states the category of finite sets and relations is Mermin–local [14], lending some credibility to our definition of Kochen–Specker contextuality. We now show a partial converse of Theorem 4.1, that is, every global section of Spec_P isolates some $x \in X$, although we do not claim that every global section is of the form $\tilde{x}$ as defined in the proof of Theorem 4.1.

Lemma 4.2. *For a set X let $\mathbf{E} = \{ q \bullet \mathrm{id}_X \mid q \in Q \}$, then $\mathbf{E}$ belongs to $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(X)$.*

We call $\mathbf{E}$, as defined in Lemma 4.2 the *trivial semialgebra on X*. Clearly there is an inclusion $\mathbf{E} \hookrightarrow \mathbf{A}$ for every $\mathbf{A}$ in $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(X)$.

Lemma 4.3. *Suppose $\mathbf{A} \subset \mathrm{Hom}(A,A)$ belongs to $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(A)$ and suppose $\mathbf{B} \subset \mathrm{Hom}(B,B)$ belongs to $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(B)$ then $\mathbf{A} \oplus \mathbf{B} \subset \mathrm{Hom}(A \sqcup B, A \sqcup B)$ belongs to $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(A \sqcup B)$.*

Lemma 4.4. *If $\mathbf{A} = e_1\mathbf{A} \oplus e_2\mathbf{A}$ belongs to $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(A)$ where e_1 is the identity morphism on some subset $E \subset A$ then $e_1\mathbf{A}$ viewed as a subsemialgebra $e_1\mathbf{A} \subset \mathrm{Hom}(E,E)$ belongs to $\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\mathrm{vN}}(E)$.*

Theorem 4.5. *Let Q be a ZDF quantale and X an object in $\mathbf{Rel}_Q$. Every global section $\chi : T \to \mathrm{Spec}_P(-)$ uniquely determines some $x \in X$.*

Proof. By Lemma 3.8 for $\mathbf{A} = \prod_i e_i\mathbf{A}$ there is one primitive idempotent element e_a such that $\chi(e_a) = 1$. For $\mathbf{B} = \prod_j d_j\mathbf{B}$ there is one d_b such that $\chi(d_b) = 1$. We claim that for e_a and d_b we have $e_a \circ d_b \neq 0$.

Let $E_a = \mathrm{supp}(e_a)$ and $E_b = \mathrm{supp}(e_b)$ and let $\mathbf{E}_1$ be the trivial semialgebra defined on the set $X \backslash (E_a \sqcup E_b)$. Let $\mathbf{E}_2$ be the trivial semialgebra on $X \backslash E_a$ and $\mathbf{E}_3$ be the trivial semialgebra on $X \backslash E_b$. Hence we have unital subsemialgebra inclusions

$$\mathbf{A} \qquad e_a\mathbf{A} \oplus d_b\mathbf{B} \oplus \mathbf{E}_1 \qquad \mathbf{B}$$
$$\searrow \qquad \nearrow \qquad \searrow \qquad \nearrow$$
$$e_a\mathbf{A} \oplus \mathbf{E}_2 \qquad d_b\mathbf{B} \oplus \mathbf{E}_3$$

By naturality, if $\chi_\mathbf{A}(e_a) = 1$ then $\chi_{e_a\mathbf{A} \oplus e_b\mathbf{B} \oplus \mathbf{E}_1}(e_a) = 1$ which implies that $\chi_{e_a\mathbf{A} \oplus e_b\mathbf{B} \oplus \mathbf{E}_1}(e_b) = 0$, which in turn implies that $\chi_\mathbf{B}(e_b) = 0$, which is a contradiction. Since there is an algebra $\mathbf{A} = \prod_{x \in X} Q$, picking a global section for this algebra amounts to picking a singleton from X. $\qquad \square$

Remark 4.6. Spekkens toy theory [26] can be modelled in $\mathbf{Rel}$ using Frobenius algebras as a notion of observable [6], and hence by Remark 2.10 can be modelled by commutative von Neumann semialgebras. In Spekkens Toy theory the *ontic states* of the physical system, which represent local hidden variables, are represented by the singleton elements of the underlying set, and hence we see a correspondence between the ontic states of the theory and the global sections in the model.

5 Topologising the State Space

The concept of the "spectrum" of an algebraic object is a broad one, appearing across many fields of mathematics: it lies at the heart of a family of deep results connecting algebra and topology [19]; it is a fundamental concept in algebraic geometry [25]; and it is central to the algebraic approach to classical physics described in Figure 1. [23]. In each case one endows the spectrum of an algebraic object with a

topology called the *Zariski topology*. Here we extend the definition of Zariski topology to the k^*–ideals of a semialgebra and to the characters on a semialgebra and hence to the prime spectrum and Gelfand spectrum of S^*–semialgebras.

Definition 5.1. Let $\mathscr{A}$ be a locally small †–symmetric monoidal category with finite †–biproducts. Let X be some object, and let $\mathbf{A}$ be an object in $\mathscr{A}$-$\mathbf{Alg}_{\mathrm{vN}}(X)$. For each ideal $J \subset \mathbf{A}$ define the sets $\mathbb{V}_P(J) = \{K \in \mathrm{Spec}_P(\mathbf{A}) \mid J \subset K \}$. We take the collection of $\mathbb{V}_P(J)$ to be a basis of closed sets for the *Zariski topology* on $\mathrm{Spec}_P(\mathbf{A})$. Consider the set $\mathrm{Spec}_G(\mathbf{A})$. For each ideal $J \subset \mathbf{A}$ define the set $\mathbb{V}_G(J) = \{\rho \in \mathrm{Spec}(\mathbf{A}) \mid J \subset \ker(\rho) \}$. We take the collection of $\mathbb{V}_G(J)$ to be a basis of closed sets for the *Zariski topology* on $\mathrm{Spec}_G(\mathbf{A})$.

Hence, under the interpretation of Figure 1. we see that our sets of states are in fact topological spaces. Recall, a space is T_0 if all points are *topologically distinguishable*, that is, for every pair of points x and y there is at least one open set containing one but not both of these points.

Theorem 5.2. *For an S^*–semialgebra $\mathbf{A}$ the Zariski topology on $\mathrm{Spec}_P(\mathbf{A})$ is compact and T_0, and for $i : \mathbf{A} \hookrightarrow \mathbf{B}$ the function $\tilde{i} : \mathrm{Spec}_P(\mathbf{B}) \to \mathrm{Spec}_P(\mathbf{A})$ in continuous with respect to this topology. For an S^*–semialgebra $\mathbf{A}$ the Zariski topology on $\mathrm{Spec}_G(\mathbf{A})$ is compact, and for $i : \mathbf{A} \hookrightarrow \mathbf{B}$ the function $\tilde{i} : \mathrm{Spec}_G(\mathbf{B}) \to \mathrm{Spec}_G(\mathbf{A})$ in continuous with respect to this topology.*

Theorem 5.2 states the the prime spectrum and Gelfand spectrum give us functors of the form

$$\mathscr{A}\text{-}\mathbf{Alg}(A)^{\mathrm{op}} \xrightarrow{\ \mathrm{Spec}_P\ } \mathbf{Top} \qquad\qquad \mathscr{A}\text{-}\mathbf{Alg}(A)^{\mathrm{op}} \xrightarrow{\ \mathrm{Spec}_G\ } \mathbf{Top}$$

Note the Gelfand spectrum need not even be T_0 in general, as we will see in Example 5.4.

Theorem 3.11 relates the prime spectrum and the Gelfand spectrum for the case when $\mathscr{A}$ is the category $\mathbf{Rel}_Q$ for a Q a ZDF quantale. The following theorem gives us an insight into the nature of this relationship in terms of the topological structure on these spectra.

Theorem 5.3. *For Q a ZDF quantale and $\mathbf{A}$ in $\mathbf{Rel}_Q$-$\mathbf{Alg}_{\mathrm{vN}}(X)$, each $\xi_{\mathbf{A}} : \mathrm{Spec}_G(\mathbf{A}) \to \mathrm{Spec}_P(\mathbf{A})$, as defined in Theorem 3.11, is a quotient of topological spaces where $\rho_1 \sim \rho_2$ iff ρ_1 and ρ_2 are not distinguishable by the Zariski topology.*

The map $\xi_{\mathbf{A}}$ identifies those characters which have the same kernel, which are precisely those characters which the Zariski topology on $\mathrm{Spec}_G(\mathbf{A})$ cannot distinguish.

Theorem 5.3 allows us to think of $\mathrm{Spec}_P(\mathbf{A})$ as a coarse–graining of the state space $\mathrm{Spec}_G(\mathbf{A})$ of our physical system. To illustrate this we revisit Example 3.12.

Example 5.4. Let $\mathbf{A}$ be as in Example 3.12. The Zariski topology on $\mathrm{Spec}_P(\mathbf{A})$ has a basis consisting of the sets $\{J_1, J_2\}$, $\{K_1, K_2\}$, $\{J_1\}$, and $\{K_1\}$. It is easy to check that this topology is T_0 but that it is not T_1 and therefore not Hausdorff. For $\mathrm{Spec}_G(\mathbf{A})$ the Zariski topology has a basis of closed sets $\{\varphi_1, \varphi_2, \varphi_3\}$, $\{\varphi_1\}$, $\{\theta_1, \theta_2, \theta_3\}$, $\{\theta_1\}$. It is easy to check that there is no open set distinguishing φ_2 and φ_3 from one another, nor θ_2 from θ_3 as these respective pairs of characters have the same kernels and hence $\mathrm{Spec}_G(\mathbf{A})$ fails even to be T_0. Note that $\xi_{\mathbf{A}}(\varphi_2) = \xi_{\mathbf{A}}(\varphi_3) = K_1$ and $\xi_{\mathbf{A}}(\theta_2) = \xi_{\mathbf{A}}(\theta_3) = J_1$, and hence the topologically indistinguishable points are identified by $\xi_{\mathbf{A}}$.

References

[1] Abramsky, S., Barbosa, R.S., Kishida, K., Lal, R., Mansfield, S.: Contextuality, cohomology and paradox. In: Logic in Computer Science. pp. 211–228 (2015)

[2] Abramsky, S., Brandenburger, A.: A unified sheaf-theoretic account of non-locality and contextuality. CoRR abs/1102.0264 (2011), `http://arxiv.org/abs/1102.0264`

[3] Abramsky, S., Coecke, B.: A categorical semantics of quantum protocols. In: Symposium on Logic in Computer Science. pp. 415–425 (2004)

[4] Abramsky, S., Heunen, C.: H* -algebras and nonunital Frobenius algebras: First steps in infinite dimensional categorical quantum mechanics, vol. 71, pp. 14–37. American Mathematical Society (2012)

[5] Bohr, N.: Discussion with Einstein on epistemological problems in atomic physics. In: Schilpp, P.A. (ed.) The Library of Living Philosophers, Albert Einstein: Philosopher-Scientist, vol. 7. Open Court (1949)

[6] Coecke, B., Edwards, B.: Toy quantum categories (extended abstract). In: Electr. Notes Theor. Comput. Sci. vol. 270, pp. 29–40 (2011)

[7] Coecke, B., Pavlovic, D., Vicary, J.: A new description of orthogonal bases. In: Mathematical Structures in Computer Science. vol. 23, pp. 555–567 (2013)

[8] Conway, J.B.: A Course in Operator Theory, Graduate Studies in Mathematics, vol. 21. American Mathematical Society (2000)

[9] Doering, A., Isham, C.: What is a thing? In: Coecke, B. (ed.) New Structures in Physics, chap. 13, pp. 753–940. Springer, Heidelberg (2011)

[10] Dunne, K.: A new perspective on observables in the category of relations: a spectral presheaf for relations. In: Quantum Interaction (2016)

[11] Dunne, K.: On the structure of abstract H*–algebras. In: Quantum Physics and Logic (2017)

[12] Flori, C.: A First Course in Topos Quantum Theory. Springer–Verlag (2013)

[13] Givant, S., Halmos, P.: Introduction to Boolean Algebras. Springer–Verlag (2009)

[14] Gogioso, S., Zeng, W.: Mermin non-locality in abstract process theories. In: EPTCS. vol. 195, pp. 228–246 (2015)

[15] Golan, J.S.: The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science. Longman Group UK Limited (1992)

[16] Heunen, C.: Semimodule enrichment. In: Electr. Notes Theor. Comput. Sci. vol. 218, pp. 193–208 (2008)

[17] Isham, C., Butterfield, J.: A topos perspective on the Kochen-Specker theorem: I. quantum states as generalised valuations. `arXiv:quant-ph/9803055`

[18] Isham, C.J.: Lectures on Quantum Theory: Mathematical and Structural Foundations. Imperial College Press (1995)

[19] Johnstone, P.T.: Stone Spaces. Cambridge University Press (1982)

[20] Kelly, G.M., Laplaza, M.L.: Coherence for compact closed categories. In: Journal of Pure and Applied Algebra. vol. 19, pp. 193–213 (1980)

[21] Kochen, S., Specker, E.P.: Logical structures arising in quantum theory. In: The Logico-Algebraic Approach to Quantum Mechanics. pp. 263–276 (1975)

[22] Mitchell, B.: Theory of Categories. New York Academic Press (1965)

[23] Nestruev, J.: Smooth Manifolds and Observables, Graduate Texts in Mathematics, vol. 220

[24] Rosenthal, K.I.: Quantales and their applications. Pitman research notes in mathematics series, Longman Scientific & Technical (1990)

[25] Smith, J.R.: Introduction to Algebraic Geometry. Five Dimensions Press (2014)

[26] Spekkens, R.W.: Evidence for the epistemic view of quantum states. In: Phys. Rev. A. vol. 75 (2007)