

Frobenius structures over Hilbert C*-modules

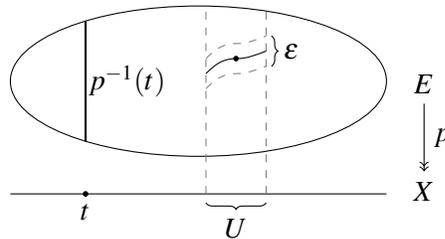
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Categorical quantum mechanics provides a powerful graphical calculus for quantum theory. It achieves this by stripping the traditional Hilbert space model of much detail. Nevertheless, the main examples remain based on Hilbert spaces, and relations between sets. The latter can be extended to take scalars in arbitrary quantales. This presentation extends scalars in the former from complex numbers to arbitrary commutative C*-algebras. In other words, we study the monoidal category of *Hilbert modules* over a commutative C*-algebra. This provides a genuinely new model, that is interesting for various reasons.

- Just like commutative C*-algebras $C_0(X)$ are dual to locally compact Hausdorff spaces X , we prove that Hilbert modules are equivalent to bundles of Hilbert spaces over locally compact Hausdorff spaces (Theorem 4.7).



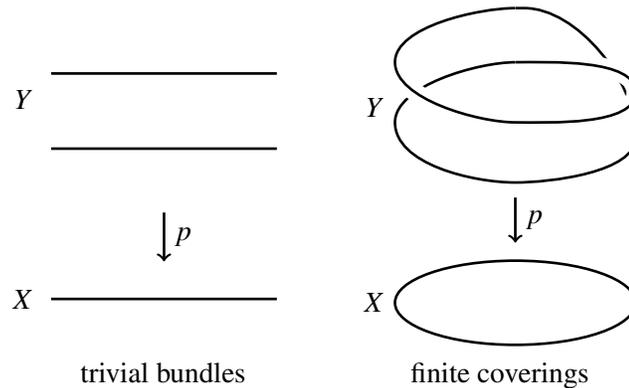
This gives a very naive model of *algebraic quantum field theory*: instead of a single Hilbert space of states, we may have Hilbert spaces $p^{-1}(t)$ over every point t of a base space X , that vary continuously with t .

- We show that the monoidal unit contains more structure than perhaps previously thought. The abstract *scalars* in the monoidal dagger category of Hilbert modules over a commutative C*-algebra $C_0(X)$ correspond to the *-algebra $C_b(X)$ (Lemma 3.1). The *subjects* of the monoidal unit $C_0(X)$ correspond to clopen subsets of X (Lemma 3.3). The category has *kernels*, but only dagger kernels when X is totally disconnected, or when one restricts to the full subcategory of dual objects (Propositions 10.1, 10.2, and 10.3).
- Letting the base space vary gives a bicategory of Hilbert bimodules, which forms an infinite continuous extension of the finite *higher-categorical* approach to categorical quantum mechanics (Proposition A.4).

Our main focus is on *Frobenius structures*, which model classical information flow and algebras of observables. For most results below we restrict to locally compact Hausdorff spaces X that are paracompact; this captures all possible (topological) manifolds.

- We prove that a Hilbert module has a (dagger) *dual object* if and only if it is finitely presented projective (Theorem 5.5). In terms of bundles of Hilbert spaces, this means that the dimension of the fibres is finite and bounded.

- We prove that specialisable dagger Frobenius structures correspond to finite-dimensional C^* -algebras that vary continuously over the base space, or more precisely, uniformly finite-dimensional fields of C^* -algebras (Theorem 7.7). In fact, we show that this correspondence of objects extends to both $*$ -homomorphisms and *completely positive maps* as morphisms (Theorem 7.10). In other words, we identify the result of applying the CP^* -construction to the category of Hilbert modules.
- We completely characterize the *commutative* dagger Frobenius structures. First, any dagger Frobenius structure is a dagger product of a nondegenerate one and a zero one. Nondegenerate specialisable dagger Frobenius structures are equivalent to *finite coverings*, i.e. continuous surjections $p: Y \rightarrow X$ such that every point of X has a neighbourhood whose preimage is a finite disjoint union of homeomorphic copies (Theorem 8.6). Let us distinguish two cases:



In the category of Hilbert spaces, i.e. if X is a singleton set, only the first case can occur. Trivial bundles are a special case of finite coverings: a finite covering is merely locally trivial.

In particular, Frobenius structures in a category like that of Hilbert modules need not copy classical information elementwise as previously thought: there may be *no copyable states* at all, even though the whole category is well-pointed. This more intricate structure should inform notions of classicality and contextuality.

- We prove a transitivity theorem, that reduces studying specialisable dagger Frobenius structures to studying *commutative* ones and *central* ones (Theorem 9.7). We give examples of each that are nontrivial, in the sense that they do not arise from Frobenius structures on Hilbert spaces.
- The category of Hilbert modules captures infinite dimension, with entirely standard methods, and without dropping unitality: although dagger Frobenius structures form local algebras of observables that are finite-dimensional, globally they can form any homogeneous C^* -algebra.

All of this shows that the category of Hilbert modules is an exciting new model where categorical quantum mechanics can take place. It opens the door to studying algebraic quantum field theory and relativistic quantum information theory using methods entirely internal to categorical quantum mechanics; the paper “*Space in monoidal categories*” gives a proof of concept of the latter. This presentation lays the foundations of the category of Hilbert modules and discusses its Frobenius structures, going toward the former.