Bayes + Hilbert = Quantum Mechanics

Alessio Benavoli   Alessandro Facchini   Marco Zaffalon
IDSIA
Lugano, Switzerland
alessio@idsia.ch   alessandro.facchini@idsia.ch   zaffalon@idsia.ch

Quantum mechanics (QM) is based on four main axioms, which were derived after a long process of trial and error. The motivations for the axioms are not always clear and even to experts the basic axioms of QM often appear counter-intuitive. In a recent paper [1], we have shown that:

- It is possible to derive quantum mechanics from a single principle of self-consistency or, in other words, that QM laws of Nature are logically consistent;
- QM is just the Bayesian theory generalised to the complex Hilbert space.

To obtain these results we have generalised the *theory of desirable gambles* (TDG) to complex numbers. TDG was originally introduced by Williams, and later reconsidered by Walley, to justify in a subjective way a very general form of probability theory.

[Theory of desirable gambles] In classical subjective, or Bayesian, probability, there is a well-established way to check whether the probability assignments of a certain subject, whom we call Alice, about the result of an uncertain experiment is valid, in the sense that they are self-consistent. The idea is to use these probability assignments to define odds—the inverses of probabilities—about the results of the experiment (e.g., Head or Tail in the case of a coin toss); and then show that there is no way to make Alice a sure loser in the related betting system, that is, to make her lose money no matter the outcome of the experiment. Historically this is also referred to as the impossibility to make a Dutch book or that the assessments are *coherent*; and Alice in these conditions is regarded as a rational subject. De Finetti [3] showed that Kolmogorov's probability axioms can be derived by imposing the principle of coherence alone on a subject's odds about an uncertain experiment.

Williams and Walley [8, 7] have later shown that it is possible to justify probability in a simpler and more elegant way. Their approach is also more general than de Finetti's, because coherence is defined purely as logical consistency without any explicit reference to probability (which is also what allows coherence to be generalised to other domains, such as quantum mechanics); the idea is to work in the dual space of gambles. To understand this framework, we consider an experiment whose outcome ω belongs to a certain space of possibilities Ω (e.g., Head or Tail). We can model Alice's beliefs about ω by asking her whether she accepts engaging in certain risky transactions, called *gambles*, whose outcome depends on the actual outcome of the experiment. Mathematically, a gamble is a bounded real-valued function on Ω, $g : \Omega \to \mathbb{R}$, which is interpreted as an uncertain reward in a linear utility scale. If Alice accepts a gamble g, this means that she commits herself to receive $g(\omega)$ *utiles*[1] if the outcome of the experiment eventually happens to be the event $\omega \in \Omega$. Since $g(\omega)$ can be negative, Alice can also lose utiles. Therefore Alice's acceptability of a gamble depends on her knowledge about the experiment.

The set of gambles that Alice accepts—let us denote it by $\mathscr{K}$—is called her set of *desirable gambles*. We say that a gamble g is *positive* if $g \neq 0$ and $g(\omega) \geq 0$ for each $\omega \in \Omega$. We say that g is *negative* if $g \neq 0$ and $g(\omega) \leq 0$ for each $\omega \in \Omega$. $\mathscr{K}$ is said to be *coherent* when it satisfies the following minimal requirements:[2]

[1] Abstract units of utility, indicating the satisfaction derived from an economic transaction.
[2] For an example see [1, Example 1].

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D1 Any positive gamble g must be desirable for Alice ($g \in \mathscr{K}$), given that it may increase Alice's capital without ever decreasing it (**accepting partial gain**).
D2 Any negative gamble g must not be desirable for Alice ($g \notin \mathscr{K}$), given that it may only decrease Alice's capital without ever increasing it (**avoiding partial loss**).
D3 If Alice finds g and h to be desirable ($g,h \in \mathscr{K}$), then also $\lambda g + \nu h$ must be desirable for her ($\lambda g + \nu h \in \mathscr{K}$), for any $0 < \lambda, \nu \in \mathbb{R}$ (**linearity of the utility scale**).

In spite of their simple character, these axioms alone define a very general theory of probability. De Finetti's (Bayesian) theory is the particular case obtained by additionally imposing some regularity (continuity) requirement and especially completeness, that is, the idea that a subject should always be capable of comparing options [7, 8].In this case, probability is derived from $\mathscr{K}$ via (mathematical) *duality*.

[QM] In [1] we have extended desirability to QM. To introduce this extension, we first have to define what is a gamble in a quantum experiment and how the payoff for the gamble is computed. To this end, we consider an experiment relative to an n-dimensional quantum system and two subjects: the gambler (Alice) and the bookmaker. The n-dimensional quantum system is prepared by the bookmaker in some quantum state. We assume that Alice has her personal knowledge about the experiment (possibly no knowledge at all).

1. The bookmaker announces that he will measure the quantum system along its n orthogonal directions and so the outcome of the measurement is an element of $\Omega = \{\omega_1, \ldots, \omega_n\}$, with ω_i denoting the elementary event "detection along i". Mathematically, it means that the quantum system is measured along its eigenvectors, i.e., the projectors and ω_i is the event "indicated" by the i-th projector.
2. Before the experiment, Alice declares the set of gambles she is willing to accept. Mathematically, a gamble G on this experiment is a Hermitian matrix in $\mathbb{C}^{n\times n}$; the space of all Hermitian $n \times n$ matrices is denoted by $\mathbb{C}_h^{n\times n}$.
3. By accepting a gamble G, Alice commits herself to receive $\gamma_i \in \mathbb{R}$ utiles if the outcome of the experiment eventually happens to be ω_i. The value γ_i is defined from G and Π^* as follows $\Pi_i^* G \Pi_i^* = \gamma_i \Pi_i^*$ for $i = 1, \ldots, n$. It is a real number since G is Hermitian.

The subset of all positive semi-definite and non-zero (PSDNZ) matrices in $\mathbb{C}_h^{n\times n}$ constitutes the set of *positive gambles*, whereas the set of negative gambles is similarly given by all gambles $G \in \mathbb{C}_h^{n\times n}$ such that $G \lneq 0$. Alice examines the gambles in $\mathbb{C}_h^{n\times n}$ and comes up with the subset $\mathscr{K}$ of the gambles that she finds desirable. Alice's rationality is then characterised by simply applying the rational axioms of the theory of desirable gambles to the space of hermitian matrices:

S1 Any PSDNZ (positive gamble) G must be desirable for Alice ($G \in \mathscr{K}$), given that it may increase Alice's utiles without ever decreasing them (**accepting partial gain**).
S2 Any $G \lneq 0$ (negative gamble) must not be desirable for Alice ($G \notin \mathscr{K}$), given that it may only decrease Alice's utiles without ever increasing them (**avoiding partial loss**).
S3 If Alice finds G and H desirable ($G,H \in \mathscr{K}$), then also $\lambda G + \nu H$ must be desirable for her ($\lambda G + \nu H \in \mathscr{K}$), for any $0 < \lambda, \nu \in \mathbb{R}$ (**linearity of the utility scale**).

From a geometric point of view, a coherent set of desirable gambles $\mathscr{K}$ is a convex cone without its apex and that contains all PSDNZ matrices (and thus it is disjoint from the set of all matrices such that $G \lneq 0$). We may also assume that $\mathscr{K}$ satisfies the following additional property:

S4 if $G \in \mathscr{K}$ then either $G \gneq 0$ or $G - \varepsilon I \in \mathscr{K}$ for some strictly positive real number ε (**openness**).

This property is not necessary for rationality, but it is technically convenient as it precisely isolates the kind of models we use in QM (as well as in classical probability) [1]. The openness condition (S4) has a gambling interpretation too: it means that we will only consider gambles that are *strictly* desirable for

Alice; these are the gambles for which Alice expects gaining something—even an epsilon of utiles. For this reason, $\mathscr{K}$ is called set of *strictly desirable gambles* (SDG) in this case.

An SDG is said to be *maximal* if there is no larger SDG containing it. In [1, Theorem IV.4], we have shown that maximal SDGs and density matrices are one-to-one. The mapping between them is obtained through the standard inner product in $\mathbb{C}_h^{n\times n}$, i.e., $G \cdot R = Tr(G^\dagger R)$ with $G, R \in \mathbb{C}_h^{n\times n}$ via a *representation theorem* [1, Theorem IV.4].

This result has several consequences. First, it provides a gambling interpretation of the first axiom of QM on density operators. Second, it shows that density operators are *coherent*, since the dual of ρ is a valid SDG. This also implies that QM is self-consistent—a gambler that uses QM to place bets on a quantum experiment cannot be made a partial (and, thus, sure) loser. Third, the first axiom of QM on $\mathbb{C}_h^{n\times n}$ is structurally and formally equivalent to Kolmogorov's first and second axioms about probabilities on $\mathbb{R}^n$ [1, Sec. 2]. In fact, they can be both derived via duality from a coherent set of desirable gambles on $\mathbb{C}_h^{n\times n}$ and, respectively, $\mathbb{R}^n$. In [1] we have also derived *Born's rule* and the *other three axioms of QM as a consequence of rational gambling on a quantum experiment* and show that that **measurement, partial tracing and tensor product** are equivalent to the probabilistic notions of **Bayes' rule, marginalisation and independence.** Finally, as an additional consequence of the aforementioned representation result, in [2] we have shown that a subject who uses dispersion-free probabilities to accept gambles on a quantum experiment can always be made a sure loser: she loses utiles no matter the outcome of the experiment. We say that dispersion-free probabilities are incoherent, which means that they are logically inconsistent with the axioms of QM. Moreover, we have proved that it is possible to derive a stronger version of Gleason's theorem that holds in any finite dimension (hence even for $n = 2$), through much a simpler proof, which states that all *coherent* probability assignments in QM must be obtained as the trace of the product of a projector and a density operator.

A list of relevant bibliographic references, as well as a comparison between our approach and similar approaches like QBism [4] and Pitowsky's quantum gambles [6], can be found in [1].

References

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