A complete characterisation of All-versus-Nothing arguments on stabilisers

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Introduction

Since the formulation of classic no-go theorems by Bell [4] and Kochen-Specker [12], contextuality has gained great relevance in the development of quantum information and computation. This key characteristic feature of quantum mechanics represents one of the most valuable resources at our disposal to break through the limits of classical computation and information processing, with various applications e.g. in quantum computation speed-up [11, 16] and in device-independent quantum security [10]

Of particular interest is the notion of *strong contextuality* [3], which was originally shown to arise in quantum mechanics by Greenberger, Horne, Shimony, and Zeilinger (GHSZ) [8, 9]. In 1990, Mermin presented a simpler proof of this phenomenon, which rests on deriving an inconsistent system of equations in $\mathbb{Z}_2$, and became known in the literature as the original "All-vs-Nothing" (AvN) argument [14]. Recent work on the mathematical structure of contextuality [3] allowed a powerful formalisation and generalisation of Mermin's proof to a large class of examples in quantum mechanics using stabiliser theory [2]. We take advantage of this framework in conjunction with the graph state formalism to prove that every AvN argument for an n-qubit stabiliser state can be reduced to an AvN proof for three qubits which are local Clifford-equivalent to the tripartite GHZ state. This result is achieved through the proof of the *AvN triple theorem*, previously conjectured in [1], which provides a combinatorial characterisation of AvN arguments for stabilisers. This new description is used to develop a computational method to identify all the AvN arguments in $\mathbb{Z}_2$ on general n-qubit stabiliser states. We also present new insights into the relationship between contextuality and logical paradoxes.

1 All-vs-Nothing Arguments

Mermin's original formulation Consider a tripartite measurement scenario where each party $i = 1,2,3$ can perform Pauli measurements in $\{X_i, Y_i\}$ on the GHZ state $1/\sqrt{2}(|000\rangle + |111\rangle)$ with outcomes in $\{0,1\}$.[1] It can be shown that the possible joint outcomes must satisfy the following equations:

$$X_1 \oplus X_2 \oplus X_3 = 1 \qquad Y_1 \oplus X_2 \oplus Y_3 = 0$$
$$X_1 \oplus Y_2 \oplus Y_3 = 0 \qquad Y_1 \oplus Y_2 \oplus X_3 = 0.$$

[1] It is convenient to relabel $+1, -1, \times$ as $0, 1, \oplus$ respectively. The eigenvalues of a joint measurement $A_1 \otimes A_2 \otimes A_3$ are the products of eigenvalues at each site, so they are also ± 1 Thus, joint measurements are still dichotomic and only distinguish joint outcomes up to parity. This is still true for general n-partite scenarios.

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This system does not admit any solution, which means that there is no deterministic assignment of values to each measurement consistent with the events deemed possible by the empirical model. We conclude that the model is strongly contextual.

The stabiliser world A general n-partite measurement scenario can be modelled by the Pauli n-group P_n and its local action on the Hilbert space $H_n := (\mathbb{C}^2)^{\otimes n}$ of n-qubit states. To any subgroup $S \subseteq \mathsf{P}_n$ we associate its stabiliser V_S, i.e. the vector space of states stabilised by the elements of S.[2] Every subgroup S gives rise to an empirical model where the n parties can perform joint measurements contained in S on a state in the stabiliser V_S. This allows one to generalise Mermin's AvN argument to stabiliser states.

Because of footnotes 1 and 2, given a $P \in \mathsf{P}_n$ and a state $|\psi\rangle \in H_n$ stabilised by P, the possible joint outcomes of the joint measurement described by P must satisfy $\bigoplus_{i=1}^n P_i = 0$ if the phase of P is $+1$, or $\bigoplus_{i=1}^n P_i = 1$ if the phase of P is -1. Therefore, to any subgroup S of P_n we can associate an XOR theory $\mathbb{T}_S^\oplus$ constituted by all the equations defined above. We say that S is **AvN** if $\mathbb{T}_S^\oplus$ is inconsistent and we have the following result from [2].

Proposition 1. *Any* AvN *subgroup of* P_n *gives rise to a strongly contextual empirical model admitting an AvN argument.*

Galois connections and relations with logic There is a well-known correspondence between the rank of a subgroup S of P_n and the dimension of its stabiliser V_S [6]:

$$\text{rank } S = k \quad \Leftrightarrow \quad \dim V_S = 2^{n-k}. \tag{1}$$

We formalise and generalise this link by introducing the following Galois connection:

$$\begin{array}{ccc} \mathbf{SubGrp}(\mathsf{P}_n) & \longleftrightarrow & \mathbf{SubSp}(H_n) \\ S & \longmapsto & V_S \\ \bigcap_{v \in V} (\mathsf{P}_n)_v & \longleftarrow\!\shortmid & V. \end{array} \tag{2}$$

where $(\mathsf{P}_n)_v$ denotes the isotropy group of $v \in V$. The relation (1) can be recovered as a mere consequence of the fact that this Galois connection is tight in the sense of [15]. This formal description of the link between subgroups of P_n and their stabilisers allows us to establish a relation between (2) and the Galois connection between syntax and semantics in logic [18]:

$$\begin{tikzcd} \mathbf{SubGrp}(\mathsf{P}_n) \arrow[r, shift left] \arrow[r, phantom, "\perp", yshift=-8pt] \arrow[d, "\mathbb{T}^\oplus"'] & \mathbf{SubSp}(H_n) \arrow[l, shift left] \arrow[d, "\mathbb{M}_\oplus := \perp \circ \mathbb{T}_\oplus \circ \perp"] \\ \mathbf{Theories} \arrow[r, shift left] \arrow[r, phantom, "\perp", yshift=8pt] & \mathscr{P}(\mathbf{Struct}) \arrow[l, shift left] \end{tikzcd}$$

where $\mathbb{T}_\oplus$ is the function mapping a subgroup S to its XOR theory $\mathbb{T}_S^\oplus$. In particular it can be shown that the pair of maps $\langle \mathbb{T}^\oplus, \mathbb{M}_\oplus \rangle$ constitutes a monomorphism of adjunctions [13] which maps AvN empirical models to logical paradoxes.

[2] Note that subgroups of P_n which stabilise non-trivial subspaces must be commutative, and only contain elements with global phase ± 1.

2 Characterising AvN Arguments

The general theory of AvN arguments described above raises the natural question of whether it is possible to idenfity the stabiliser states admitting this type of proof of contextuality. A sufficient condition for the existence of an AvN argument is obtained through the notion of *AvN triple* [2]. An AvN triple is a triple $\langle e, f, g\rangle$ of elements of P_n which satisfy special combinatorial properties. We have the following result from [2]:

Theorem 2. *Any subgroup S of P_n generated by an AvN triple is AvN.*

The AvN triple theorem and its consequences We prove the converse of Theorem 2, which leads to the formulation of the *AvN triple theorem*:

Theorem 3 (AvN triple theorem)**.** *A subgroup S of P_n is AvN if and only if it contains an AvN triple.*

The theorem is proved for *graph states*, which are special types of multi-qubit states that can be represented by a graph. It is known from [17] that any stabiliser state is local-Clifford (LC) equivalent to a graph state. Since strong contextuality is preserved under LC operators, this is enough to characterise AvN arguments on stabilisers. From the proof of Theorem 3 we immediately obtain another interesting result:

Theorem 4. *Every AvN argument on an n-partite stabiliser state can be reduced to an AvN proof for a three-qubit subsystem which is LC-equivalent to the GHZ state.*

In other words, not only is the GHZ state, up to LU-equivalence, the only tripartite stabiliser state admitting an AvN argument, but it actually underlies *every* AvN proof of contextuality for general n-partite stabiliser states.

3 Applications

The characterisation given by Theorem 3 allows us to translate the complex conditions needed for an AvN argument into the simple combinatorial properties of AvN triples. Thanks to this straightforward description, we can derive a closed formula for the number of possible AvN arguments on n-partite stabiliser states:

Proposition 5. *Let $n \geq 3$. The number of AvN triples in P_n is given by*

$$\sum_{k=1}^{\frac{1}{2}(n+[n])-1} \binom{n}{2k+1}\binom{k+1}{k-1} \cdot 6^{2k+1} \cdot 22^{n-2k-1},$$

where $[n] \in \mathbb{Z}_2$ denotes the parity of n.

More significantly, we can use the AvN triple theorem and the check-vector description of elements of P_n to develop a computational method to actually identify all the AvN arguments on n-partite stabiliser states for a sufficiently small n. An implementation of this method using Mathematica [19] can be found in [5], where we present the algorithm and the resulting list of all 216 AvN triples in P_3 and all 19008 AvN triples in P_4. By Theorem 3, this list generates all the possible AvN arguments for 3-qubit and 4-qubits stabiliser states.

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