Purity through factorisation

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We give a construction that identifies the collection of pure processes (*i.e.* those which are deterministic, or without randomness) within a theory containing both pure and mixed processes. Working in the framework of monoidal categories, we define a pure subcategory relative to a family of completely mixed states. This definition arises elegantly from the categorical notion of a weak factorisation system. Our construction gives the expected result in several examples, both quantum and classical.

1 Introduction

Categorical quantum mechanics models physical theories as monoidal categories: objects are interpreted as physical systems, and morphisms are interpreted as processes that take a state of one system to a state of another [1, 9, 13]. This approach captures various physical theories uniformly:

- functions between finite sets, which may be interpreted as deterministic processes between systems with finitely many states;

- relations between finite sets, which may be interpreted as nondeterministic processes between systems with finitely many states;

- stochastic matrices (those matrices with entries in the nonnegative reals, whose columns sum to 1), which may be interpreted as probabilistic processes between systems with finitely many states;

- completely positive maps between finite-dimensional Hilbert spaces, which may be interpreted as quantum processes between systems with finitely many degrees of freedom.

In these interpretations, some, but not all, processes are *pure*, in the sense that probability plays no role:

- a function is always pure;

- a relation is pure when it is (the graph of) a partial function;

- a stochastic matrix is pure when each column has one nonzero entry;

- a completely positive map is pure when it needs no ancilla.

General morphisms are interpreted as a *mixture* of pure ones. The goal of this article is to give an *operational definition* of purity. In contrast to other approaches [4, 6, 3, 5, 10, 11], our definition merely needs the structure of monoidal categories. That is, we define what it means for a morphism to be pure using only its relationships with the other morphisms in the category (so without daggers or dual objects), and without any reference to the interpretation of morphisms as being processes between systems. The key idea is that *purification* may be regarded as part of a *factorisation system*. To evidence that our definition is useful, we prove that in the above four example cases it recaptures the desired interpretation.

We start by recalling factorisation systems in Section 2, and discuss lifting properties in Section 3. Section 4 then defines purification in terms of factorisation systems, and Section 5 illustrates that this is a good notion by showing that it captures probabilistic theories, possibilistic theories, deterministic theories, and quantum theories. Finally, Section 6 gives a comparison with previous attempts at operational definitions of purity [4, 15]. We will use the diagrammatic notation [17].

2   Factorisation

Our definition of purity is inspired by Stinespring's dilation theorem [18]. This is a theorem about the category of finite dimensional Hilbert spaces and completely positive maps [7]; Stinespring's statement allowed infinite dimensional spaces, but we don't use that generality. Hilbert spaces represent quantum systems and completely positive maps represent all the physically realisable nondeterministic processes between them [16]. The pure morphisms representing deterministic processes form a symmetric monoidal subcategory. Additionally, every system has a has a *completely mixed state* represented by a completely positive map from $\mathbb{C}$ (the monoidal unit) to the Hilbert space representing that system [10, 11]. We will draw this state diagrammatically as $\perp$.

Theorem 1 (Stinespring dilation). *Any completely positive map $A \to B$ can be written in the form*

$$
\begin{array}{c}
B \\
\boxed{p} \\
{}_{A}
\end{array}
\tag{1}
$$

for some ancilla *Hilbert space C and pure morphism p.*

Proof. See [16].   □

We can reformulate the Stinespring dilation theorem (and theorems like it) in a category-theoretic way by using a version of factorisation systems [12, 2].

Definition 2. Suppose $\mathbf{C}$ is a category and $\mathscr{L}$ and $\mathscr{R}$ are collections of morphisms in $\mathbf{C}$. We say that $(\mathscr{L}, \mathscr{R})$ is a *factorisation system* if for every morphism f in $\mathbf{C}$ we have $f = r \circ l$ for some $l \in \mathscr{L}$ and $r \in \mathscr{R}$. (We do not necessarily demand that $\mathscr{L}$ and $\mathscr{R}$ are subcategories, or that they have any other properties at all.)

Stinespring's theorem says that the category of completely positive maps has a factorisation system, with $\mathscr{L}$ consisting of all maps that introduce a mixed state, and $\mathscr{R}$ consisting of all pure maps. See also [19]. Two types of factorisation system are of particular interest:

- *Orthogonal factorisation systems* are typified by the system $(\mathrm{Surj}, \mathrm{Inj})$ in **Set**; the factorisation is found by writing a function as a surjection onto its image followed by inclusion of the image into the codomain. Orthogonal factorisation systems have the property that their factorisations are unique up to a unique isomorphism on the 'middle' object through which the morphism is being factored. In fact orthogonal factorisation systems can be defined as those which have this uniqueness property in addition to the property that $\mathscr{L}$ and $\mathscr{R}$ are replete[1] subcategories [14, C.0.19].

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[1]A subcategory is *replete* if it contains (all objects and) all isomorphisms.

- *Weak factorisation systems*[2] are typified by the system $(\mathrm{Inj}, \mathrm{Surj})$ in **Set**; the factorisation is found by letting the 'middle' object be the domain along with one extra element for each point not in the image of the original function. In a weak factorisation system the factorisations are not necessarily unique, but they do satisfy a weaker property which we will give in the next section.

The Stinespring dilation is known to never be unique. So it cannot describe an orthogonal factorisation system, but might describe a weak factorisation system. In fact, we will see that it does indeed give a weak factorisation system after a slight modification, namely expanding the collection $\mathscr{L}$ to contain all isomorphisms. This might be expected from the interpretation: the introduction of an ancillary completely mixed state resembles an injection as it maps a smaller space into a larger one, so we might have predicted in advance that there would be an analogy with $(\mathrm{Inj}, \mathrm{Surj})$ but not with $(\mathrm{Surj}, \mathrm{Inj})$.

3 Lifting

Weak factorisation systems are defined in terms of the following relation. Let **C** be any category and let f and g be morphisms in **C** (there need not be any relation between their domains and codomains). Say that f and g have the *lifting property*, and write $f \boxslash g$, when any commuting square

$$
\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
f\downarrow & \overset{h}{\nearrow} & \downarrow g \\
\bullet & \longrightarrow & \bullet
\end{array}
$$

has a morphism h making the two triangles commute.

For a collection $\mathscr{A}$ of morphisms in **C** (that are not necessarily closed under composition), define $\mathscr{A}^{\boxslash}$ and ${}^{\boxslash}\mathscr{A}$ to be the collections of morphisms that have the lifting property on the right or left of all the morphisms in $\mathscr{A}$:

$$
\mathscr{A}^{\boxslash} = \{g \in \mathrm{Mor}(\mathbf{C}) | \forall f \in \mathscr{A} : f \boxslash g\},
$$
$$
{}^{\boxslash}\mathscr{A} = \{f \in \mathrm{Mor}(\mathbf{C}) | \forall g \in \mathscr{A} : f \boxslash g\}.
$$

It is easily shown that $\mathscr{A}^{\boxslash}$ and ${}^{\boxslash}\mathscr{A}$ contain all isomorphisms and are closed under composition, and are therefore replete subcategories of **C**. We also have that ${}^{\boxslash}(-)$ and $(-)^{\boxslash}$ are order reversing in the sense that if $\mathscr{A} \subseteq \mathscr{B}$ then $\mathscr{A}^{\boxslash} \supseteq \mathscr{B}^{\boxslash}$ and ${}^{\boxslash}\mathscr{A} \supseteq {}^{\boxslash}\mathscr{B}$. Furthermore $\mathscr{A} \subseteq ({}^{\boxslash}\mathscr{A})^{\boxslash}$ and $\mathscr{A} \subseteq {}^{\boxslash}(\mathscr{A}^{\boxslash})$; pairs of order reversing functions with this property are known as *Galois connections*. It follows that alternating applications of ${}^{\boxslash}(-)$ and $(-)^{\boxslash}$ eventually cease to have an effect: ${}^{\boxslash}(({}^{\boxslash}\mathscr{A})^{\boxslash}) = {}^{\boxslash}\mathscr{A}$ and $(({}^{\boxslash}\mathscr{A}^{\boxslash}))^{\boxslash} = \mathscr{A}^{\boxslash}$.

Given any collection $\mathscr{A}$ of morphisms, we can generate a pair $(\mathscr{L}, \mathscr{R})$ by letting $\mathscr{R} = \mathscr{A}^{\boxslash}$ and then $\mathscr{L} = {}^{\boxslash}\mathscr{R}$. This pair then has the property that $\mathscr{L} = {}^{\boxslash}\mathscr{R}$ and $\mathscr{R} = \mathscr{L}^{\boxslash}$. [3] Thus for any collection of morphisms $\mathscr{A}$ we can generate a pair $(\mathscr{L}, \mathscr{R})$ and then ask if this pair happens to form a factorisation system.

Definition 3. A *weak factorisation system* is a factorisation system with $\mathscr{L} = {}^{\boxslash}\mathscr{R}$ and $\mathscr{R} = \mathscr{L}^{\boxslash}$.

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[2]To avoid confusion: being a weak factorisation system isn't weaker than being a factorisation system. It is called that merely because being weak is weaker than being orthogonal. Also note that some conventions use the words "factorisation system" for what we call an "orthogonal factorisation system".

[3]In the theory of model categories, $(\mathscr{L}, \mathscr{R})$ is called *cofibrantly generated* from $\mathscr{A}$. We won't use this terminology here.

Example 4. Let us see why $(\mathrm{Inj}, \mathrm{Surj})$ is a weak factorisation system in **Set**.

First note that this is indeed a factorisation system. Given a function $f\colon A \to B$ let i be the inclusion $A \to A \sqcup (B \smallsetminus \mathrm{im}(f))$ and let $s = f \sqcup \mathrm{id}_{B \smallsetminus \mathrm{im}(f)}\colon A \sqcup (B \smallsetminus \mathrm{im}(f)) \to \mathrm{im}(f) \sqcup (B \smallsetminus \mathrm{im}(f)) = B$. Then i is injective, s is surjective, and $s \circ i = f$.

Next we want to show that $i \boxempty s$ for any injection $i\colon A \to B$ and surjection $s\colon C \to D$. So suppose that

$$
\begin{array}{ccc}
A & \xrightarrow{\ j\ } & C \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle s} \\
B & \xrightarrow[\ k\ ]{} & D
\end{array}
$$

commutes. Define $h\colon B \to C$ by for any $b \in \mathrm{im}(i)$ defining $h(b) = j \circ i^{-1}(b)$ (which is well defined because i is injective), and for any $b \notin \mathrm{im}(i)$ defining $h(b)$ to be any element of the preimage of $k(b)$ under s (which is nonempty because s is surjective). Then $h \circ i = j$ and $s \circ h = k$, as required.

Thus $\mathrm{Inj} \subseteq {}^{\boxempty}\mathrm{Surj}$ and $\mathrm{Surj} \subseteq \mathrm{Inj}^{\boxempty}$. Finally we must prove that there are no noninjections in ${}^{\boxempty}\mathrm{Surj}$ nor nonsurjections in $\mathrm{Inj}^{\boxempty}$. Suppose $i'\colon A \to B$ is not an injection, so there is some c with $c = i'(a)$ and $c = i'(b)$. Let $i' = s \circ i$ be an $(\mathrm{Inj}, \mathrm{Surj})$-factorisation. Then there is no h making

$$
\begin{array}{ccc}
A & \xrightarrow{\ i\ } & \bullet \\
{\scriptstyle i'}\downarrow & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle s} \\
B & \xrightarrow[\ \mathrm{id}_B\ ]{} & B
\end{array}
$$

commute, because $i(a) \neq i(b)$ and h must map c to both $i(a)$ and $i(b)$.

Likewise, if $s'\colon A \to B$ is not a surjection then pick some d not in its image and let $s' = s \circ i$ be an $(\mathrm{Inj}, \mathrm{Surj})$-factorisation. Then there is no h making

$$
\begin{array}{ccc}
A & \xrightarrow{\ \mathrm{id}_A\ } & A \\
{\scriptstyle i}\downarrow & {\scriptstyle h}\nearrow & \downarrow{\scriptstyle s'} \\
\bullet & \xrightarrow[\ s\ ]{} & B
\end{array}
$$

commute, because h must map the nonempty set $s^{-1}(d)$ to the empty set $s'^{-1}(d)$.

Next, we adapt these notions to the setting of monoidal categories. The problem is that if **C** is a monoidal category then the subcategories ${}^{\boxempty}\mathscr{A}$ and $\mathscr{A}^{\boxempty}$ need not be monoidal subcategories. We adapt our definitions accordingly[4]:

$$
f \oslash g \iff \forall A, B, C, D \in \mathrm{Ob}(\mathbf{C})\colon (\mathrm{id}_A \otimes f \otimes \mathrm{id}_B) \boxempty (\mathrm{id}_C \otimes g \otimes \mathrm{id}_D)
$$

$$
\mathscr{A}^{\oslash} = \{g \in \mathrm{Mor}(\mathbf{C}) \mid \forall f \in \mathscr{A}\colon f \oslash g\}
$$
$$
{}^{\oslash}\mathscr{A} = \{f \in \mathrm{Mor}(\mathbf{C}) \mid \forall g \in \mathscr{A}\colon f \oslash g\}
$$

Now ${}^{\oslash}\mathscr{A}$ and $\mathscr{A}^{\oslash}$ are replete monoidal subcategories, and $(-)^{\oslash}$ and ${}^{\oslash}(-)$ form a Galois connection. When **C** is braided or symmetric the definition of $\oslash$ can be simplified because the ancillary objects need only be introduced on one side of f and g. So $f \oslash g$ if and only if $\forall A, C \in \mathrm{Ob}(\mathbf{C})\colon (\mathrm{id}_A \otimes f) \boxempty (\mathrm{id}_C \otimes g)$. The monoidal subcategories ${}^{\oslash}\mathscr{A}$ and $\mathscr{A}^{\oslash}$ are then braided or symmetric respectively.

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[4]The definition of $\oslash$ in terms of $\boxempty$ resembles complete positivity of quantum processes: for f to be completely positive it is not enough to preserve positivity of states, also $f \otimes \mathrm{id}_A$ must preserve positivity of states.

Definition 5. A *monoidal weak factorisation system* is a factorisation system with $\mathscr{L} = {}^{\oslash}\mathscr{R}$ and $\mathscr{R} = \mathscr{L}^{\oslash}$.

Example 6. If we consider **Set** to be a symmetric monoidal category under Cartesian products $\times$, then our example $(\mathrm{Inj}, \mathrm{Surj})$ is indeed a monoidal weak factorisation system. Since $\oslash$ is a stricter property than $\boxempty$ we have that ${}^{\oslash}\mathrm{Surj} \subseteq {}^{\boxempty}\mathrm{Surj} = \mathrm{Inj}$ and $\mathrm{Inj}^{\oslash} \subseteq \mathrm{Inj}^{\boxempty} = \mathrm{Surj}$ so it suffices to check that $i \oslash s$ for all injections i and surjections s. But this is immediate. If i is an injection then $\mathrm{id}_A \times i$ is an injection for any set A. Likewise, if s is a surjection then $\mathrm{id}_B \times s$ is a surjection for any set B. So $\mathrm{id}_A \times i \boxempty \mathrm{id}_B \times s$ for all A and B and hence $i \oslash s$.

# 4   Purity

Suppose a monoidal category **C** comes equipped with a morphism $I \to A$ for each object A. We draw these morphisms as $\def\mixed{\rotatebox{0}{\rlap{$\top$}}}$ ⊥, and call them *completely mixed states*. Call the collection of all the completely mixed states $\mathscr{M}'$.

The following definition is our operational description of purity.

Definition 7. Given a monoidal category equipped with a family of completely mixed states, define the *pure morphisms* to be those in the class $\mathscr{P} = \mathscr{M}'^{\oslash}$, and the *mixing morphisms* to be those in the class $\mathscr{M} = {}^{\oslash}\mathscr{P}$.

The interpretation of the collection $\mathscr{M}$ is not immediately clear, since the notion of mixing is already captured by the completely mixed states $\mathscr{M}'$. It will become clear from the examples below that the elements of $\mathscr{M}$ represent processes which do not create any additional purity. For example, if k is an isomorphism, then
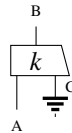
is in $\mathscr{M}$; we call such morphisms *simple mixing maps*.

Thus we have a pair $(\mathscr{M}, \mathscr{P})$, which might be a factorisation system. If it is a factorisation system, then by construction it will certainly be a monoidal weak one. Our examples will in fact have the following, even stronger, property.

Definition 8. A category **C** *has purification* if all morphisms are of the form (1) for pure p.

For example, Stinespring's theorem says precisely that the category of completely positive maps has purification. If **C** has purification then $(\mathscr{M}, \mathscr{P})$ is a factorisation system because $\mathrm{id} \otimes \mixed$ is always in $\mathscr{M}$.

Dualising the above definitions, we can also consider monoidal categories equipped with families of morphisms $\mixed$ which we think of as taking a system and discarding it. We refer to these morphisms as *discarding effects* and denote the collection of them as $\mathscr{D}'$. Call the morphisms in the class $\mathscr{C} = {}^{\oslash}\mathscr{D}'$ *copure*, and define the *discarding morphisms* to be those in the class $\mathscr{D} = \mathscr{C}^{\oslash}$. Say **C** has *copurification* if all morphisms are of the form

for some copure c.

We will see in examples that the copure morphisms represent processes that do not destroy any information, whereas destruction of information is the only effect of the discarding morphisms.

Let's decode our abstract definitions of pure and mixing from the language of weak factorisation systems into the diagrammatic notation. To simplify the diagrams we will assume **C** is symmetric. Then the morphism p is pure if and only if:

$$\cdots \quad \Longrightarrow \quad b \text{ is of the form } \cdots \text{ where } \cdots \tag{$*$}$$

and m is mixing if and only if

$$\cdots \quad \Longrightarrow \quad b \text{ is of the form } \cdots \text{ where } \cdots \tag{$\dagger$}$$

for any pure morphism p.

These two criteria are quite involved, and it is often useful to think of the following two special cases where a and b have particularly simple forms. If p is pure, then

$$\cdots \quad \Longrightarrow \quad b \text{ is of the form } \cdots \text{ where } \cdots \tag{$**$}$$

and if m is mixing, then

$$\cdots \text{ with } p \text{ pure} \quad \Longrightarrow \quad \text{there exists } h \text{ with } \cdots \text{ and } \cdots \tag{$\dagger\dagger$}$$

These special cases are necessary but not sufficient, so if p is pure it certainly obeys ($**$) and if m is mixing it certainly obeys ($\dagger\dagger$), but the converse is not necessarily true. However, in many of the particular categories **C** that we are interested in, we do in fact find that ($**$) and ($\dagger\dagger$) are as strong as ($*$) and ($\dagger$). So when it comes to calculating the classes $\mathscr{P}$ and $\mathscr{M}$ for a particular category **C**, one can often proceed by finding the classes of morphisms which obey ($**$) and ($\dagger\dagger$), and then verifying that these morphisms further satisfy ($*$) and ($\dagger$).

5   Examples

The main claim of this paper is that our definitions of 'pure', 'mixing', 'copure' and 'discarding' are in accordance with expectations when interpreting morphisms as processes: if a monoidal category **C** equipped with a family of morphisms $\bar{\mp}$ is interpreted as a collection of processes in which $\bar{\mp}$ represents a completely mixed state, then the family of morphisms which are deterministic in this interpretation should be those in the class $\mathscr{P}$. We will give four examples.

5.1 Probabilistic

Let **FStoch** be the symmetric monoidal category whose objects are nonempty[5] finite sets (with the Cartesian product as $\otimes$) and whose morphisms are matrices of nonnegative reals with the domain labelling the columns and the codomain labelling the rows (with matrix multiplication as $\circ$, and Kronecker product of matrices as $\otimes$). Interpret a matrix with entries (f_{ij}) as a process which, if it begins in state i, has probability f_{ij} of going to state j. The morphisms $\overline{\mp}$ and $\doteq$ are given by the column vector of 1s and the row vector of 1s[6].

Lemma 9. *Suppose that $\mathscr{R}$ is the collection of matrices which have at most one nonzero entry in each column, and $\mathscr{L}$ is the collection of matrices which have exactly one nonzero entry in each row and at least one nonzero entry in each column. Then $(\mathscr{L},\mathscr{R})$ is a factorisation system, and $m \oslash p$ for $m \in \mathscr{L}$, $p \in \mathscr{R}$.*

Proof. See Appendix A. □

Lemma 10. *The symmetric monoidal subcategory $\mathscr{P}$ consists of precisely those matrices which have at most one nonzero entry in each column. Dually, $\mathscr{C}$ consists of precisely those matrices which have at most one nonzero entry in each row.*

Proof. Suppose $p : X \to Y$ is pure, and suppose that $p_{y_0 x_0} \neq 0$ and $p_{y_1 x_0} \neq 0$. In $(**)$, let $b : X \times \{0,1\} \to Y$ be defined by setting $b_{yx0} = p_{yx}$ for all x,y except that $b_{y_1 x_0 0} = 0$, and $b_{yx1} = 0$ for all x,y except that $b_{y_1 x_0 1} = p_{y_1 x_0}$. Then $h_{x_0 x_0 1} \neq 0$, so $b_{y_0 x_0 1} = (p \circ h)_{y_0 x_0 1} \neq 0$. Hence $y_0 = y_1$. So p has at most one nonzero entry in each column.

Conversely, suppose that p has at most one nonzero entry in each column. Then by Lemma 9 $p \oslash (\mathrm{id}_B \otimes \overline{\mp})$, so p is pure.

The second statement follows by taking the transpose of all matrices involved. □

Lemma 11. *The symmetric monoidal subcategory $\mathscr{M}$ consists of precisely those matrices which have exactly one nonzero entry in each row and at least one nonzero entry in each column. Dually, $\mathscr{D}$ consists of precisely those matrices which have exactly one nonzero entry in each column and at least one nonzero entry in each row.*

Proof. Suppose $m : X \to Y$ is in $\mathscr{M}$. In $(\dagger\dagger)$, define $p : X \times Y \to Y$ by $p_{yxy} = m_{yx}$ and $p_{y'xy} = 0$ if $y \neq y'$. Let $y \in Y$. Then $(p \circ h)_{yy} = 1$, so there are $y' \in Y$ and $x \in X$ with $h_{xy'y} \neq 0$. Now $(h \circ m)_{xy'x} = (\mathrm{id}_X \times \overline{\mp}_Y)_{xy'x} = 1$, and so $m_{yx} \neq 0$. If also $m_{yx'} \neq 0$, then $(h \circ m)_{x'y'x} \neq 0$ and so $x' = x$. Thus m has exactly one nonzero entry in each row.

The matrix $\mathrm{id}_X \times \overline{\mp}_Y$ has at least one nonzero entry in each column. Since $h \circ m = \mathrm{id}_X \times \overline{\mp}_Y$ the map m must also have at least one nonzero entry in each column.

Conversely, suppose that a relation m has exactly one nonzero entry in each row. Then by Lemma 9 and Lemma 10 $p \oslash m$ for any pure p, so $m \in \mathscr{M}$.

The second statement follows by taking the transpose of all matrices involved. □

Theorem 12. *The category **FStoch** has purification and copurification. Hence the pairs $(\mathscr{M},\mathscr{P})$ and $(\mathscr{C},\mathscr{D})$ are monoidal weak factorisation systems.*

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[5]In each example we will remove the 0 object from the category in question because this object does not have a completely mixed state.

[6]For simplicity we will forego the usual normalisation demand that the columns sum to 1. This allows us to write the completely mixed state with entries 1 rather than $1/n$.

Proof. Any matrix $f: I \to J$ of nonnegative reals factors as follows. Let the ancilla system be the set J. Let p be the matrix with $((i,j),j')$-entry $\delta_{jj'} f_{ij}$; this is pure by Lemma 10. Then f is given by (3). The dual statement is obtained by transposing f. $\qquad\square$

5.2 Possibilistic

Let **FRel** be the symmetric monoidal category whose objects are nonempty finite sets (with the Cartesian product as $\otimes$) and whose morphisms are relations between the source and target. Interpret a relation $r: A \to B$ as a nondeterministic process that might send an $a \in A$ to any of the $b \in B$ to which it is related. We take $\doubledownarrow$ to be the relation that relates the point to everything in the target, and $\doubleuparrow$ to be the relation that relates everything in the source to the point.

Lemma 13. *Suppose that $\mathscr{R}$ is the collection of partial functions (i.e. relations in which everything in the source is related to at most one element of the target) and $\mathscr{L}$ is the collection of surjective, injective and total relations (i.e. those in which everything in the target is related to exactly one thing in the source and everything in the source is related to at least one thing in the target). Then $(\mathscr{L}, \mathscr{R})$ is a factorisation system, and $m \oslash p$ for $m \in \mathscr{L}$, $p \in \mathscr{R}$.*

Proof. See Appendix A. $\qquad\square$

Lemma 14. *The symmetric monoidal subcategory $\mathscr{P}$ consists precisely of the partial functions. Dually, $\mathscr{C}$ consists precisely of the injective relations (i.e. those in which everything in the target is related to at most one element of the source).*

Proof. Suppose $p \in \mathscr{P}$, and $(x_0, y_0), (x_0, y_1) \in p$. In (**), set $b = \{(0,x,y) \mid (x,y) \in p, (x,y) \neq (x_0,y_1)\} \cup \{(1,x_0,y_1)\}$. Then $(1,x_0,x_0) \in h$, so $(1,x_0,y_0) \in p \circ h = b$. Hence $y_0 = y_1$. Thus p is a partial function.
  Conversely, suppose that p is a partial function. Then by Lemma 13 $p \oslash (\mathrm{id}_B \otimes \doubledownarrow)$, so p is pure.
  The second statement follows by taking the dual of all relations involved. $\qquad\square$

Lemma 15. *The symmetric monoidal subcategory $\mathscr{M}$ consists of precisely the surjective, injective and total relations. Dually, $\mathscr{D}$ is precisely the surjective total functions (i.e. the relations in which everything in the source is related to exactly one thing in the target and everything in the target is related to at least one thing in the source).*

Proof. Suppose $m: X \to Y$ is in $\mathscr{M}$. In (††), set $p = \{(x,y,y) \mid (x,y) \in m\}$. Let $y \in Y$. Then $(y,y) \in p \circ h$, so there are $y' \in Y$ and $x \in X$ with $(y,x,y') \in h$. Now $(x,x,y') \in \mathrm{id}_X \times \doubledownarrow_Y = h \circ m$, and so $(x,y) \in m$. If also $(x',y) \in m$, then $(x',x,y') \in h \circ m = \mathrm{id}_X \times \doubledownarrow_Y$ and so $x' = x$. Thus m is surjective and injective.
  The relation $\mathrm{id}_X \times \doubledownarrow_Y$ is total. Since $h \circ m = \mathrm{id}_X \times \doubledownarrow_Y$ the map m must also be total.
  Conversely, suppose that a relation m is surjective and injective. Then by Lemma 13 and Lemma 14 $p \oslash m$ for any pure p, so $m \in \mathscr{M}$.
  The second statement follows by taking dual relations. $\qquad\square$

Theorem 16. *The category **FRel** has purification and copurification. Hence the pairs $(\mathscr{M}, \mathscr{P})$ and $(\mathscr{C}, \mathscr{D})$ are monoidal weak factorisation systems.*

Proof. Any relation $r: A \to B$ factors as the introduction of a completely mixed state on B, given by the relation $A \to A \times B$ that relates $a \in A$ to (a,b) for any $b \in B$, followed by a partial function $A \times B \to B$ that sends (a,b) to b when $(a,b) \in r$. Dually, r factors as an injective relation $A \to A \times B$ that relates $a \in A$ to (a,b) if $(a,b) \in r$, followed by discarding A, i.e. the relation $A \times B \to B$ that relates (a,b) to b. $\qquad\square$

5.3 Deterministic

Let **FSet** be the category whose objects are nonempty finite sets (with the Cartesian product as $\otimes$) and whose morphisms are functions. Interpret a function $f\colon A \to B$ as a deterministic process that sends $a \in A$ to $f(b) \in B$. Take $\doteqdot$ to be the function that maps everything in its source to the point (in fact we had to chose this function, since I is terminal). There is no notion of a completely mixed state in this category.

Lemma 17. *The symmetric monoidal subcategory $\mathscr{C}$ consists precisely of the injections.*

Proof. Unfolding the definitions, $f\colon A \to B$ is in $\mathscr{C}$ when for all finite sets C, D, E and functions g, h making the following square commute there is a fill-in k:

$$
\begin{array}{ccc}
C \times A & \xrightarrow{\;g\;} & E \times D \\
{\scriptstyle \mathrm{id}_C \times f}\big\downarrow & \overset{k}{\nearrow} & \big\downarrow {\scriptstyle \pi_1} \\
C \times B & \xrightarrow[\;h\;]{} & E
\end{array}
$$

If this is the case, taking $D = A$, $E = C = 1$, $g = \mathrm{id}$, and $h = \pi_1$ shows that f has a left-inverse and so is injective. For the converse we fix $d \in D$. When f is injective, we can define $k(c, b) = g(c, a)$ if $f(a) = b$, and $k(c, b) = (d, h(c, b))$ otherwise. $\qquad\square$

Lemma 18. *The symmetric monoidal subcategory $\mathscr{D}$ consists precisely of the surjections.*

Proof. Unfolding definitions with the previous lemma, $g\colon C \to D$ is in $\mathscr{D}$ when for all finite sets A, B, E, F, injections $f\colon A \rightarrowtail B$, and functions h, k making the following square commute there is a fill-in:

$$
\begin{array}{ccc}
E \times A & \xrightarrow{\;h\;} & F \times C \\
{\scriptstyle \mathrm{id}_E \times f}\big\uparrow\!\!\big\downarrow & \overset{l}{\nearrow} & \big\downarrow {\scriptstyle \mathrm{id}_F \times g} \\
E \times B & \xrightarrow[\;k\;]{} & F \times D
\end{array}
$$

If this is the case, taking $B = D$ and $A = E = F = 1$ shows that g has a right-inverse and so is surjective. Conversely, if g is surjective, then $g \in \mathscr{C}^{\boxtimes}$ by Example 4, so certainly $g \in \mathscr{D} = \mathscr{C}^{\oslash}$. $\qquad\square$

The previous two lemmas prove that $(\mathscr{C}, \mathscr{D})$ is just $(\mathrm{Inj}, \mathrm{Surj})$, our canonical example of a weak factorisation system. Example 6 showed that it is also a monoidal weak factorisation system.

Theorem 19. *The category **FSet** has copurification.*

Proof. Any function $f\colon A \to B$ factors as the injection $A \rightarrowtail B \times A$ given by $a \mapsto (f(a), a)$, followed by the discarding map $\pi_1\colon B \times A \to B$. $\qquad\square$

5.4 Quantum

Let **Quant** be the category of finite-dimensional nonzero Hilbert spaces and completely positive maps. Let $\doteqdot$ be the state with density matrix given by the identity, and let $\doteqdot$ take the trace. There is a canonical functor $F\colon \mathbf{FHilb} \to \mathbf{Quant}$, and the pure morphisms in **Quant** are usually defined to be the ones in the image of this functor [16, 8]. We will show that our definitions agree with this.

Lemma 20. *(a) Every completely positive map has a* minimal *Stinespring dilation p as in* (1)*, such that any other Stinespring dilation p′ allows a coisometry[7] j ∈ **FHilb** satisfying:*

$$\text{(diagram)} \qquad = \qquad \text{(diagram)} \tag{2}$$

(b) A Stinespring dilation p as in (1) *is minimal if and only if the following morphism is a surjection:*

$$\text{(diagram)}$$
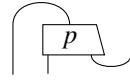
(c) If p and p′ are Stinespring dilations as in (1) *of the same map, and their ancillas C and C′ satisfy* dim(C) ≤ dim(C′)*, then there is a coisometry j satisfying* (2)*.*

Proof. See Appendix A. □

Lemma 21. *The symmetric monoidal subcategory* $\mathscr{P}$ *consists of precisely those morphisms of the form* $F(f)$ *for* $f \in \mathrm{Mor}(\mathbf{FHilb})$*.*

Proof. First we assume p pure and show it is in the image of F. Let P be a Stinespring dilation of p. Without loss of generality, assume p nonzero. As p is pure, $(**)$ gives h with:

$$\text{(diagram)} \quad = \quad \text{(diagram)} \qquad \text{and} \qquad \text{(diagram)} \quad = \quad \text{(diagram)}$$

Let H be a Stinespring dilation of h. By Lemma 20(b) the identity is clearly a minimal one. So

$$\text{(diagram)} \quad = \quad | \qquad \text{implies} \qquad \text{(diagram)} \quad = \quad \text{(diagram)}$$

for some coisometry i. Therefore:

$$\text{(diagram)} \quad = \quad \text{(diagram)} \quad = \quad \text{(diagram)} \quad = \quad \text{(diagram)}$$

Now pick any state $a\colon \mathbb{C} \to A$ in **FHilb** that makes $i \circ (a \otimes {\equiv})$ nonzero. Any scalar in **Quant** lies in the image of F, as do P and a. Hence so does:

$$\text{(diagram)} \quad \left(\text{(diagram)} \right)^{-1} \quad = \quad \text{(diagram)}$$

---

[7] A linear map between two Hilbert spaces is a *coisometry* if and only if $j \circ j^\dagger = \mathrm{id}$. Its image in **Quant** satisfies $j \circ {\equiv} = {\equiv}$.

For the converse, we assume $p \in \mathbf{FHilb}$ and prove it pure. Suppose $(\mathrm{id} \otimes p) \circ a = b \circ (\mathrm{id} \otimes \doteqdot)$. We want to find some h satisfying $(*)$. Without loss of generality we may assume b nonzero. If A and B are Stinespring dilations of a and b, then:

$$\begin{array}{ccc} \boxed{\begin{array}{c} p \\ A \\ \bot_{C} \end{array}} & = & \boxed{\begin{array}{c} B \\ {}_{C'}\bot \bot_{C''} \end{array}} \end{array}$$

Pick such A and B with the further property that $\dim(C) \le \dim(C')\dim(C'')$. This can be done by picking any A and B and then enlarging C'' by applying an arbitrary coisometry from a space with dimension greater than $\dim(C)/\dim(C')$. Then Lemma 20(c) provides a coisometry j satisfying $(\mathrm{id} \otimes p) \circ A \circ (\mathrm{id} \otimes j) = B$. Define $h = A \circ (\mathrm{id} \otimes j) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \doteqdot)$. Then

$$\begin{array}{ccccccc} \boxed{\begin{array}{c} p \\ h \end{array}} & = & \boxed{\begin{array}{c} p \\ A \\ j \end{array}} & = & \boxed{\begin{array}{c} B \end{array}} & = & \boxed{\begin{array}{c} b \end{array}} \end{array}$$

and

$$\begin{array}{ccccccc} \boxed{\begin{array}{c} h \end{array}} & = & \boxed{\begin{array}{c} A \\ j \end{array}} & = & \boxed{\begin{array}{c} A \end{array}} & = & \boxed{\begin{array}{c} a \end{array}} \end{array}$$

as required. □

Lemma 22. *The symmetric monoidal subcategory $\mathscr{M}$ consists of only the simple mixing maps.*

Proof. Let $m \in \mathscr{M}$, and let M be a minimal Stinespring dilation of m. We will exhibit an inverse of M. By Lemma 21, M is pure, so $(\dagger\dagger)$ provides h satisfying:

$$\begin{array}{ccccc} \boxed{\begin{array}{c} h \\ m \end{array}} & = & \Big| \; \doteqdot & \quad \text{and} \quad & \boxed{\begin{array}{c} M \\ h \end{array}} = \Big| \end{array}$$

Let H be a Stinespring dilation of h.

$$\begin{array}{ccccc} \boxed{\begin{array}{c} H \\ M \end{array}} & = & \Big| \; \doteqdot & \quad \text{and} \quad & \boxed{\begin{array}{c} M \\ H \end{array}} = \Big| \end{array}$$

By Lemma 20(b) the right hand side of each of these equations is a minimal Stinespring dilation, so there are coisometries j and j' satisfying:

$$\begin{array}{ccccc} \boxed{\begin{array}{c} H \\ M \end{array}} & = & \Big| \; \boxed{j} & \quad \text{and} \quad & \boxed{\begin{array}{c} M \\ H \end{array}} = \Big| \; \boxed{j'} \end{array}$$

Hence

$$\text{[diagram]}$$

Lemma 20(b) lets us remove M to see $j = \text{id} \otimes j'$. It follows that $H \circ (\text{id} \otimes j'^{\dagger})$ is the inverse of M.   □

Since the category of completely positive maps is self-dual in a way that fixes the inclusion of **FHilb**, the collection of copure morphisms is precisely the same as the collection of pure ones. This coincidence is special to the quantum case. The symmetric monoidal subcategory $\mathscr{D}$ consists precisely of the simple discarding maps.

Theorem 23. *The category* **Quant** *has purification and copurification. Hence the pairs* $(\mathscr{M}, \mathscr{P})$ *and* $(\mathscr{C}, \mathscr{D})$ *are monoidal weak factorisation systems.*

Proof. Purification follows from Lemma 21 and Theorem 1, copurification by taking duals.   □

6   Related work

Chiribella [4] defines a map p to be pure if:

$$\text{[diagram]} \qquad \Longrightarrow \qquad b \text{ is of the form } \text{[diagram]}$$

Selby and Coecke [15] define a map p to be pure if:

$$\text{[diagram]} \qquad \Longrightarrow \qquad \exists h, h' : \quad \text{[diagram]} \quad \text{and} \quad \text{[diagram]} \quad \text{and} \quad \text{[diagram]}$$

Both definitions yield the same pure subcategory in **Quant**, but they disagree in the other cases. For example, consider **FRel**. According to $(*)$ the pure morphisms in **FRel** are the partial functions. According to Selby and Coecke's definition the collection of pure morphisms is smaller; only the partial injections. The collection is smaller yet under Chiribella's definition; only the relations which relate at most one pair of elements. So these definitions produce a stricter notion of purity than that of this paper.

Another difference is that these definitions define purity in relation to the discarding map $\doteq$ rather than the completely mixed state $\doteq$. This means that the resulting collection of pure maps often bares some relation to the maps which we would call copure. This can also be seen in **FRel**; the maps which are pure under Selby and Coecke's definition are precisely those in $\mathscr{P} \cap \mathscr{C}$.

Finally, neither of these two definitions guarantees that the collection of pure maps forms a monoidal subcategory. We have already seen this in the case of Chiribella's definition, which says that the identity morphisms are not pure in **FRel**. To see this for Selby and Coecke's definition consider the category given by the usual order on the positive integers, with $\times$ as the monoidal product and the morphism $(n \geq 1)$ as $\doteq$. Then a morphism $(n \geq m)$ is pure if and only if $m > n/2$. Such morphisms are not closed under $\circ$ or $\otimes$. Of course $\mathscr{P}$ as defined by $(*)$ is always a monoidal category, because it is of the form $\mathscr{A}^{\oslash}$.

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A Proofs

Proof of Lemma 9. Any matrix $f : I \to J$ of nonnegative reals factors as follows. The matrix represented by $\mathrm{id}_I \otimes {=}_J$ has exactly one nonzero entry in each row and at least one nonzero entry in each column.

Let $p : I \times J \to J$ be the matrix with $((i,j),j')$-entry $\delta_{jj'} f_{ij}$; this has at most one nonzero entry in each column. Then f is given by:

$$\tag{3}$$

So $(\mathscr{L}, \mathscr{R})$ is a factorisation system. It remains to prove that $m \oslash p$ for $m \in \mathscr{L}$, $p \in \mathscr{R}$. It suffices to show $m \boxdot p$, because if $p \in \mathscr{L}$ then $\mathrm{id}_A \times p$ is also in $\mathscr{L}$, and similarly for $m \in \mathscr{R}$. If $b \circ m = p \circ a$, then we can factor a and b and look for h making the diagram below left commute. So it suffices to prove that any commuting square on the right below has such an h.

$$
\begin{array}{ccccc}
\bullet & \xrightarrow{m_a} & \bullet & \xrightarrow{p_a} & \bullet \\
{\scriptstyle m}\downarrow & & \uparrow{\scriptstyle h} & & \downarrow{\scriptstyle p} \\
\bullet & \xrightarrow{m_b} & \bullet & \xrightarrow{p_b} & \bullet
\end{array}
\qquad
\begin{array}{ccc}
W & \xrightarrow{m'} & Y \\
{\scriptstyle m}\downarrow & \nearrow{\scriptstyle h} & \downarrow{\scriptstyle p'} \\
X & \xrightarrow{p} & Z
\end{array}
$$

Define h as follows. For each $y \in Y$ with $p'_{zy} = 0$ for all $z \in Z$, take the unique $w \in W$ with $m'_{yw} \neq 0$ and pick $x \in X$ with $m_{xw} \neq 0$, then set $h_{yx} = m'_{yw}/m_{xw}$. Now for each $w \in W$ and $z \in Z$ say $x \in X_{wz} \subseteq X$ if $m_{xw} \neq 0$ and $p_{zx} \neq 0$, and similarly $y \in Y_{wz} \subseteq Y$ if $m'_{yw} \neq 0$ and $p'_{zy} \neq 0$. Then since $pm = p'm'$,

$$\sum_{x \in X_{xz}} p_{zx} m_{xw} = \sum_{y \in Y_{xz}} p'_{zx} m'_{xw}.$$

Call this quantity c_{wz}, and for each $x \in X_{wz}$, $y \in Y_{wz}$ define $h_{yx} = p_{zx} m'_{yw}/c_{wz}$. Let $h_{yx} = 0$ if its value has not already been defined. Then $hm = m'$ and $p'h = p$, as required. $\qquad\square$

Proof of Lemma 13. Any relation $r : A \to B$ factors as the introduction of a completely mixed state on B, given by the relation $A \to A \times B$ that relates $a \in A$ to (a,b) for any $b \in B$, followed by a partial function $A \times B \to B$ that sends (a,b) to b when $(a,b) \in r$. Since the introduction of a completely mixed state is injective, surjective and total, we have that $(\mathscr{L}, \mathscr{R})$ is a factorisation system.

It remains to prove that $m \oslash p$ for $m \in \mathscr{L}$, $p \in \mathscr{R}$. By the same logic as used in the proof of Lemma 9 it suffices to prove that any commuting square

$$
\begin{array}{ccc}
W & \xrightarrow{m'} & Y \\
{\scriptstyle m}\downarrow & \nearrow{\scriptstyle h} & \downarrow{\scriptstyle p'} \\
X & \xrightarrow{p} & Z
\end{array}
$$

has such an h. Let h be the relation containing $(x,y) \in X \times Y$ if both x and y are related to the same $w \in W$ and either x and y are related to the same $z \in Z$ or y is not related to any $z \in Z$. $\qquad\square$

Proof of Lemma 20. Parts (a) and (b) are well-known [19]. To prove (c) it is easier to work in the dual case. Suppose $(\mathrm{id} \otimes \stackrel{\centerdot}{=}) \circ p = (\mathrm{id} \otimes \stackrel{\centerdot}{=}) \circ p'$. Then (a) gives a minimal Stinespring dilation $\tilde{p}$ with ancilla $\tilde{C}$ and isometries i and i' such that $p = (\mathrm{id} \otimes i) \circ \tilde{p}$ and $p' = (\mathrm{id} \otimes i') \circ \tilde{p}$. It suffices to exhibit an isometry $i'' : C \to C'$ with $i'' \circ i = i'$. To construct i'', note that the space C can be written as $\mathrm{im}(i) \oplus \mathrm{im}(i)^{\perp}$ and the space C' can be written as $\mathrm{im}(i) \oplus \mathrm{im}(i')^{\perp}$. Since $\mathrm{im}(i)$ and $\mathrm{im}(i')$ are isometric copies of $\tilde{C}$, there is a canonical unitary u between them. Pick an isometry $\mathrm{im}(i)^{\perp} \to \mathrm{im}(i')^{\perp}$ (which exists since $\dim(\mathrm{im}(i')^{\perp}) \geq \dim(\mathrm{im}(i)^{\perp}))$ and let i'' be the direct sum of u and this isometry. $\qquad\square$